The 2^k Factorial Design

Introduction

- Factorial Designs are widely used in experiments involving several factors.
- There are several special cases of the general factorial design that are important because they are widely used, and form the basis of other designs of considerable practical value.
- The most important of these special cases is that of k factors at only two levels each, called a 2^k factorial design.

Introduction

- 2^k designs are particularly useful in the early stages of experimental work when they are many factors to be investigated.
- It provides the smallest number of runs with which k factors can be studied.

The 2^2 design

- The first design in the series is one with only two factors, say A and B, each at two levels.
- This design is called a 2^2 factorial design.
- The levels of the factors may be called ‘low’ and ‘high’.

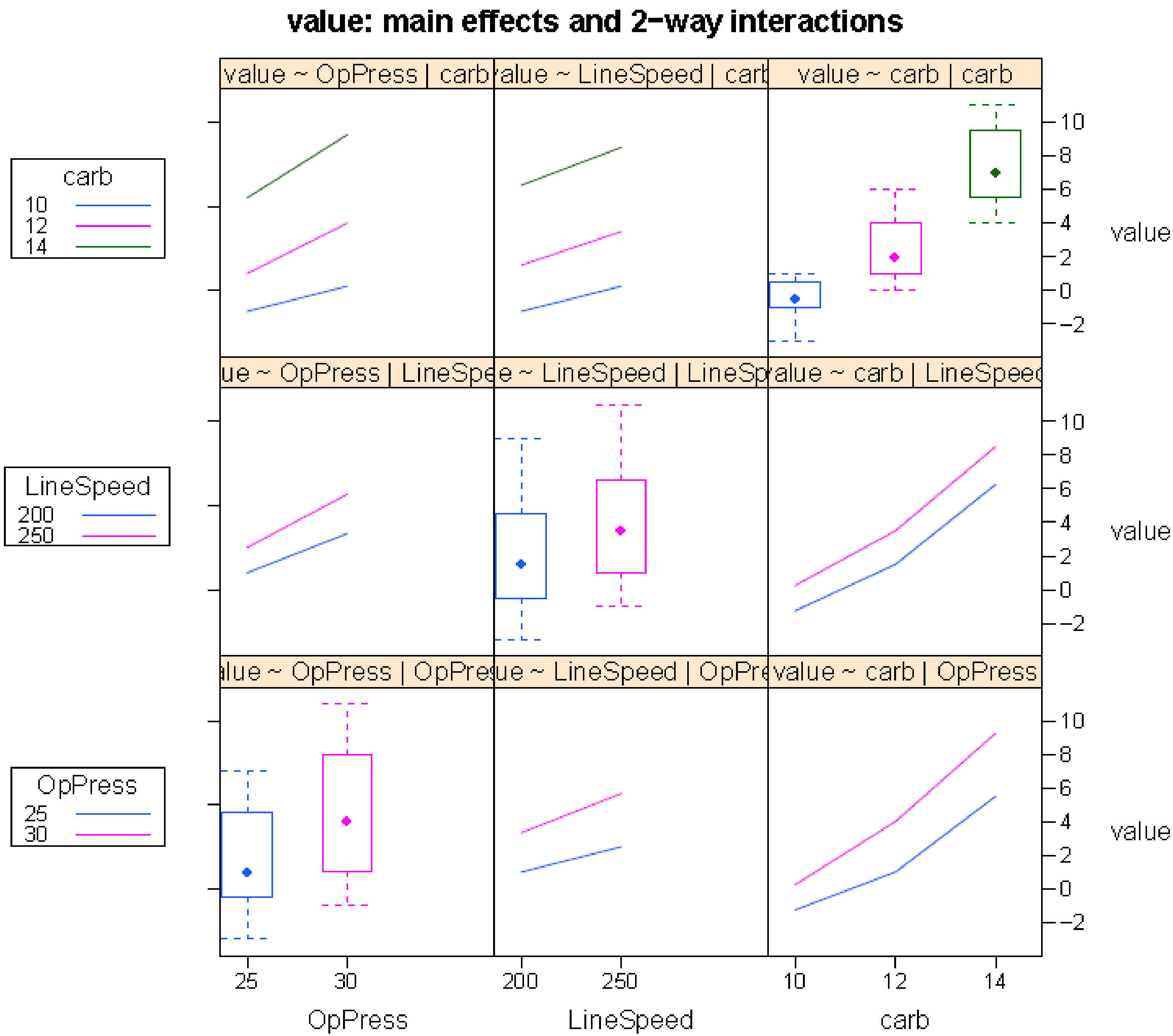
Example

- The effect of percent carbonation (10% or 12%) and operating pressure (25 psi or 30 psi) on the fill height of a carbonated beverage (Pepsi).
- Let percent carbonation be factor A, and operating pressure be factor B.
- The experiment is replicated three times.

Data

Treatment Combination	Replicate			
	I	II	III	Total
A low, B low	28	25	27	80 (1)
A high, B low	36	32	32	100 a
A low, B high	18	19	23	60 b
A high, B high	31	30	29	90 ab

Note that the high level of any factor is denoted by the corresponding letter and the low level of a factor is denoted by the absence of the corresponding letter.



value: main effects and 2-way interactions
value ~ OpPress | carb
value ~ LineSpeed | carb
value ~ carb | carb
ue ~ OpPress | LineSpe
e ~ LineSpeed | LineSp
value ~ carb | LineSpeed
lue ~ OpPress | OpPre
ue ~ LineSpeed | OpPre
value ~ carb | OpPress
carb
10
12
14
LineSpeed
200
250
OpPress
25
30
-2
0
2
4
6
8
10
value
25
30
OpPress
200
250
LineSpeed
10
12
14
carb

Main Effects and Interactions

- We define the effect of a factor as the change in response produced by a change in the level of that factor.

  $A = \frac{1}{2n}\{[ab - b] + [a - (1)]\}$

  $= \frac{1}{2n}[ab + a - b - (1)]$

- Similarly,

  $B = \frac{1}{2n}[ab + b - a - 1]$

  $AB = \frac{1}{2n}[ab + 1 - a - b]$

Main Effects and Interactions

- Using the example data, we may estimate the average effects as:

  A = 50/6

  B = - 30/6

  AB = - 10/6

- The effect of A is positive, the effect of B is negative, and the interaction appears to be small relative to the two main effects.
- ANOVA can be used to confirm this interpretation.

ANOVA

Table 5-14 Analysis of Variance for Example 5-3

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0	P-Value
Percentage of carbonation (A)	252.750	2	126.375	178.412	<0.0001
Operating pressure (B)	45.375	1	45.375	64.059	<0.0001
Line speed (C)	22.042	1	22.042	31.118	0.0001
AB	5.250	2	2.625	3.706	0.0558
AC	0.583	2	0.292	0.412	0.6713
BC	1.042	1	1.042	1.471	0.2485
ABC	1.083	2	0.542	0.765	0.4867
Error	8.500	12	0.708		
Total	336.625	23			

The 2^3 Design

- Suppose that three factors, A, B, and C, each at two levels, are of interest. The design is called a 2^3 factorial design.
- There are eight treatment combinations written in standard order as: (1), a, b, ab, c, ac, bc, abc.
- There are seven degrees of freedom; one degree of freedom associated with each main effect and interaction: A, B, C, AB, AC, BC, ABC.

The General 2^k Design

- A 2^k design includes k main effects, $\binom{k}{2}$ two factor interactions, $\binom{k}{3}$ three factor interactions, …., and one k factor interaction.
- The same notation is used for treatment combinations. For example: in a 2^5 design abd denotes A, B, D, at the high level; and C, E at the low level.
- Treatment combinations may be written in standard order. For example, in a 2^4 design:
- To estimate an effect, we can use a table of plus and minus signs.

A Single Replicate of the 2^k Design

- Even for a moderate number of factors, the total number of treatment combinations in a 2^k factorial design is large. Example: 2^6 design.
- Frequently, available resources only allow a single replicate of the design to be run, called an unreplicated factorial.
- With only one replicate, there is no estimate for error.

A Single Replicate of the 2^k Design

- To analyze an unreplicated factorial, we assume that certain higher order interactions are negligible and combine their mean squares to estimate the error.
- Sparsity of effects principal: most systems are dominated by some of the main effects and low order interactions, and most high order interactions are negligible.

Additional Concepts in Factorial Designs

The 3^k Factorial Design

- The 3^k Factorial Design is a factorial arrangement with k factors each at three levels.
- We refer to the three levels of the factors as low (0), intermediate (1), and high (2).
- For example, in a 3^2 design, the nine treatment combinations are denoted by 00, 01, 10, 02, 20, 11, 12, 21, 22.

The 3^k Factorial Design

- The 3^k factorial design is considered by experimenters who are concerned about curvature in the response.
- The addition of a third level allows the relationship between the response and each factor to be modeled with a quadratic relationship.
- Other alternatives:
  - response surface designs
  - 2^k design augmented with center points

Example: 3^2 Design

- The simplest 3^k factorial design is the 3^2 design, which has two factors, each at three levels.
- The $3^2 = 9$ treatment combinations are: 00, 01, 10, 02, 20, 11, 12, 21, 22.
- There are eight degrees of freedom between these nine treatment combinations: the main effects A and B have 2 degrees of freedom each, and the AB interaction has 4 degrees of freedom.

Example: 3^2 Design

- When a factor has three levels, it will have two degrees of freedom.
- Therefore, the associated sums of squares can be broken down into two components: one that represents the linear effect (SSA_L) and the other that represents the quadratic effect (SSA_Q).
- A linear effect is where the value of the response variable changes at almost a constant rate over the different levels.
- A quadratic effect is where the value of the response variable changes along the lines of a quadratic relationship.