Factorial Designs

Definitions and Principles

- Many experiments involve the study of the effects of two or more factors. *Factorial designs* are most efficient for this type of experiment.
- In a factorial design, all possible combinations of the levels of the factors are investigated in each replication.
- If there are *a* levels of factor A, and *b* levels of factor B, then each replicate contains all *ab* treatment combinations.

Main Effects

- The main effect of a factor is defined to be the change in response produced by a change in the level of a factor.
- The main effect of A is the difference between the average response at A_1 and A_2

		FACTOR B	
		B_1	B_2
FACTOR A	A_1	20	30
	A_2	40	52

Interaction

- In some experiments we may find that the difference in response between the levels of one factor is not the same at all levels of the other factor. When this occurs, there is an *interaction* between the factors.
- At B_1, the A effect is:
- At B_2, the A effect is:

		FACTOR B	
		B_1	B_2
FACTOR A	A_1	20	40
	A_2	50	12

Interaction graphs

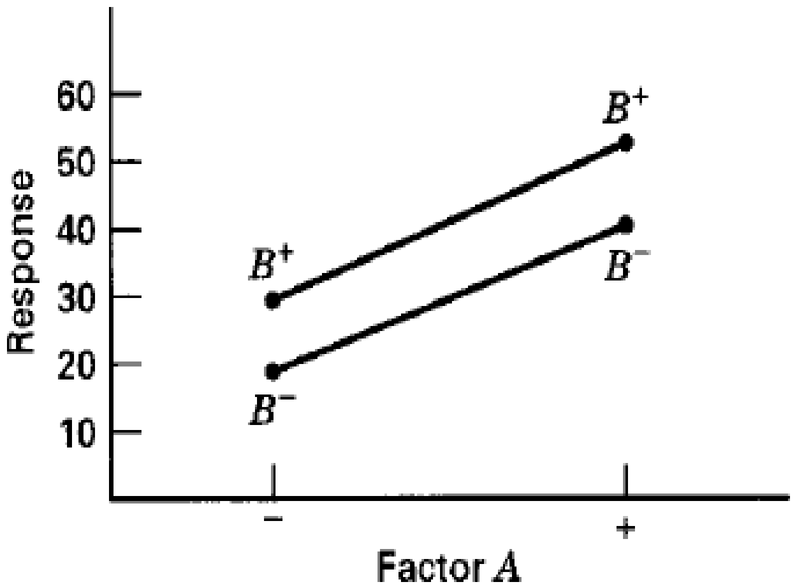

Figure 5-3 A factorial experiment without interaction.

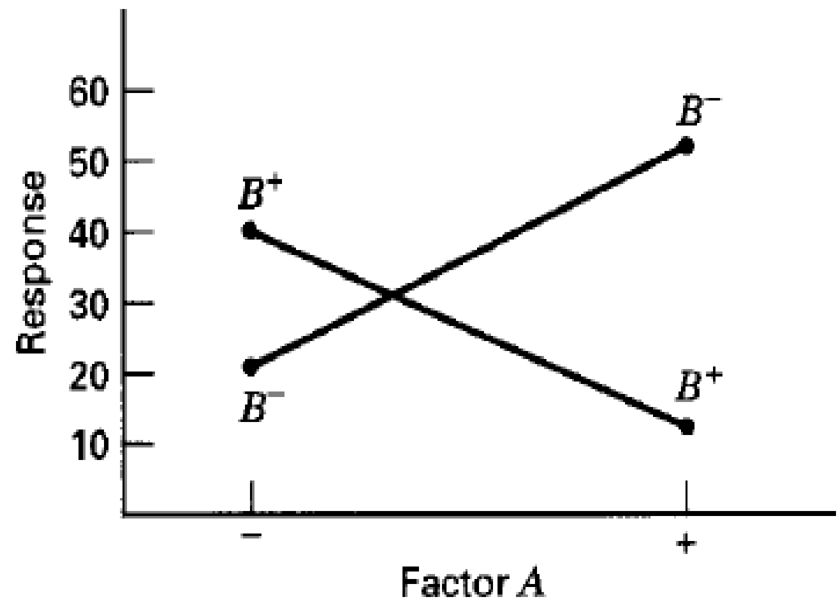

Figure 5-4 A factorial experiment with interaction.

Notes

- Graphs are frequently useful in interpreting significant interactions.
- When an interaction is large, the main effects have little practical meaning.
- A significant interaction will often *mask* the significance of main effects.

Advantages of Factorials

- They are more efficient than one-factor-at-a-time experiments.
- A factorial design is necessary when interactions may be present to avoid misleading conclusions.
- Factorial designs allow the effects of a factor to be estimated at several levels of the other factors, yielding conclusions that are valid over a range of experimental conditions.

The Two-Factor Factorial Design

- The simplest type of factorial designs involve only two factors or sets of treatments.
- There are *a* levels of factor A, and *b* levels of factor B, and each replicate contains all *ab* treatment combinations.
- In general, there are *n* replicates.

Example

- An engineer tests 3 plate materials for a new battery at 3 temperature levels (15°F, 70°F, and 125°F).
- Four batteries (replicates) are tested at each combination of plate material and temperature, and all 36 tests are run in random order.
- Questions:

  1. What effects do material type and temperature have on the life of a battery?

  2. Is there a material that would give long life *regardless of temperature?*

Data

Table 5-4 Life Data (in hours) for the Battery Design Experiment

Material Type	Temperature (°F) 15			70			125			$y_{i..}$
1	130	155	(539)	34	40	(229)	20	70	(230)	998
	74	180		80	75		82	58		
2	150	188	(623)	136	122	(479)	25	70	(198)	1300
	159	126		106	115		58	45		
3	138	110	(576)	174	120	(583)	96	104	(342)	1501
	168	160		150	139		82	60		
$y_{.j.}$	1738			1291			770			$3799 = y_{...}$

ANOVA

Table 5-5 Analysis of Variance for Battery Life Data

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0	P-Value
Material types	10,683.72	2	5,341.86	7.91	0.0020
Temperature	39,118.72	2	19,559.36	28.97	0.0001
Interaction	9,613.78	4	2,403.44	3.56	0.0186
Error	18,230.75	27	675.21		
Total	77,646.97	35			

Material type-temperature plot

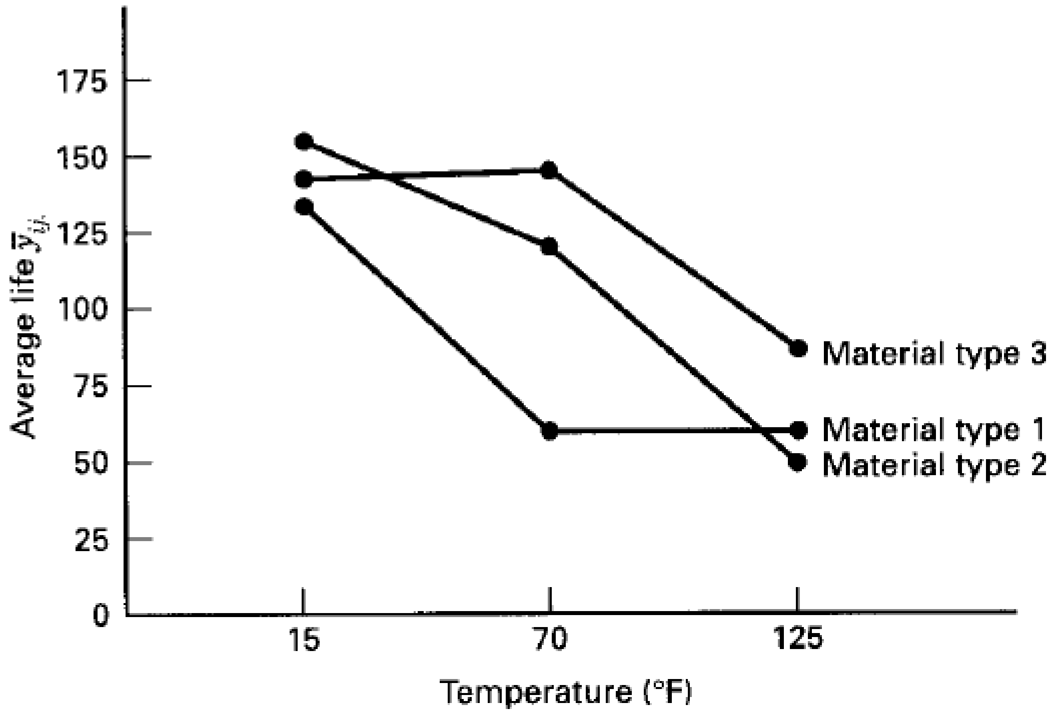

Figure 5-9 Material type–temperature plot for Example 5-1.

No Interaction analysis

Table 5-8 Analysis of Variance for Battery Life Data Assuming No Interaction

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
Material types	10,683.72	2	5,341.86	5.95
Temperature	39,118.72	2	19,559.36	21.78
Error	27,844.52	31	898.21	
Total	77,646.96	35		

Material types: p-value= 0.00636

Temperature: p-value= 1.24 e-6

Thoughts on
Factorial vs One-Factor Designs

- Consider 2 factors A and B, each with 2levels

	B1	B2
A1	20	40
A2	50	12

Interaction

- Main effect A:
  - over the 2 levels of B the A effect is different:
    - 30 vs -28
    - It depends on the level chosen for the other factor, B

Thoughts

- By use of the factorial design, the interaction can be estimated, as the AB treatment combination
- In the 1-factor design, can only estimate main effects A and B
- The same 4 observations can be used in the factorial design, as in the 1-factor design, but gain more information (e.g. on the interaction)

Conclusions

- The factorial design is *more efficient* than the 1-factor
- A factorial design is *necessary*, when interactions are present, to avoid a misleading conclusion
- Estimation of one factor at different levels of the other factor could yield conclusions over a *range* of conditions for the experiment

Additional Concepts in Factorial Designs

Random Effects and

Degrees of Freedom

Fixed and Random Effects

Fixed Effect:

- the levels of a factor are pre-determined
- the inference will be made only on the levels used in the experiment

Random Effect:

- the levels of a factor are randomly chosen
- the inference will be drawn about a population, from which the factors are chosen

Example:

A chemist studies the effect of analytical labs on the chemical analysis of a substance.

- any lab in a population is of interest, therefore a few are chosen at random

Degrees of Freedom

Example :

Three treatments, each at 3 doses, for reducing blood pressure are compared. Two trials are run.

2 Factors:

treatments (3) and doses (3) : 9 combinations

2 trials : 2 runs

Total : 9x2 = 18 combinations with 17 df

Note:

Comparison with *1-factor* analysis

- the 9 combinations could be regarded as the 9 levels of 1 factor ; 8 df
- DF :

- Treatment combinations	8
- Error	9
- Total	17

Only 1 F-test in the ANOVA

Degrees of Freedom

If analyzed as a 1-factor design, are the treatments and doses separately estimated?

In the 3^2 – factorial, df for each factor are:

Combinations		8
Treatment	2	
Dose	2	
Interaction	4	
Error		9
Total		17

Have, therefore, 3 F-tests in the ANOVA

Fixed or Random Effects for treatment and dose?