Model Uncertainty, Ambiguity Aversion and Implications for Catastrophe Insurance Market

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Abstract

Uncertainty is the inherent nature of catastrophe models because people usually have incomplete knowledge or inaccurate information in regard to such rare events. In this paper we study the decision makers’ aversion to model uncertainty using various entropy measures proposed in microeconomic theory and statistical inferences theory. Under the multiplier utility preference framework and in a static Cournot competition setting, we show that reinsurers’ aversion to catastrophe model uncertainty induces a cost effect resulting in disequilibria, i.e., limited participation, an equilibrium price higher than the actuarially fair price, and possible market break-down in catastrophe reinsurance markets. We show that the tail behavior of catastrophic risks can be modeled by a generalized Pareto distribution through entropy minimization. A well-known fact about catastrophe-linked securities is that they have zero or low correlations with other assets. However, if investors’ aversion to model uncertainty is higher than some threshold level, adding heavy-tailed catastrophic risks into the risk portfolio will have a negative diversification effect.

JEL Classification: C0, C1, D8, G0, G2

Key words: Model Uncertainty, Ambiguity Aversion, Catastrophe Insurance Market, Entropy, Generalized Pareto Distribution
1. Introduction

Catastrophe risk refers to the low-frequency and high-severity risk due to natural catastrophic events, such as hurricane, earthquake and floods, or man-made catastrophic events, such as terrorism attacks. There is an increasing trend for the associated impacts of catastrophe risks to insurance and reinsurance industry. For instance, natural disasters costed the global insurance industry roughly USD 110 billion in 2011, making it the second most expensive year only to year 2005 (Swiss Re 2012a).

Because of rareness of catastrophe events, there is very limited historical data that can be used to estimate the distribution of such risks. Alternatively, the insurance industry heavily relies on catastrophe models that simulate a huge number of probable catastrophic events. Although significant advancements in catastrophe modeling have been made through years, the prediction of catastrophic events and their associated impacts remain an “imprecise science” (Guy Carpenter, 2011). From the modeling perspective, catastrophe models are generally subject to two main sources of uncertainties: one is related to the randomness of natural hazard events, the other is the uncertainty due to incomplete information and knowledge of the hazard (Gross and Windeler, 2005). Incomplete knowledge, lack of historical data, and assumptions introduced from different disciplines involved in the modeling process can all introduce uncertainties to catastrophe models and lead to inaccurate estimate. For instance, on March 11, a magnitude 9.0 earthquake struck northeastern Japan, followed by a devastating tsunami. However, the economic losses were largely due to the effects of the tsunami rather than the ground shaking. Even though tsunami models are widely used in scientific and engineering communities, they have never been explicitly integrated by the insurance industry in its earthquake models (Swiss Re 2012a). And the costs associated with the liquefaction risks and business interruption losses
are largely underestimated in current earthquake models (Swiss Re 2012b). It is not rare that competing catastrophe models will yield materially different estimation for the same risk. For instance, the estimated two standard error interval (i.e. 68% confidence interval) for a national writer’s 100-year or higher probable maximum loss (PML) yielded by U.S. hurricane models goes between 60% and 190% of the estimated PML. And for localized writers, this interval can go from 25% and 430% of the PML estimate (Guy Carpenter, 2011).

Catastrophe models play an important role in the catastrophe insurance market, because the market participants, such as insurers, reinsurers, brokers, rating agencies, and institutional investors of insurance-linked securities, have utilized them for various purposes including rate making, risk management, credit rating and insurance-linked security pricing. However, the existing evidence suggests that catastrophe models are subject to incomplete knowledge and information, and the estimators yielded by those models are not precise. Decision makers who rely on those models are, in turn, exposed to model uncertainty. Several questions arise naturally: How will decision makers, such as reinsurers, respond to the catastrophe model uncertainty? How will their decisions, such as risk taking, be affected if they exhibit aversion to model uncertainty? What are the possible consequences of model uncertainty to the catastrophe insurance market?

The economically meaningful distinction between risk and uncertainty has been well recognized in economic literature. Roughly speaking, risk refers to the randomness due to a known probability distribution and uncertainty refers to the situation where the probability distribution is unknown. Ellsberg (1961) and following field experiments reveal that confidence, knowledge, the amount of information, and the confidence with the underlying probability distribution are important for decision makers and decision makers' aversion to uncertainty is
different from aversion to risk. It is particularly true for the catastrophe insurance market where insurers are constantly struggling for the unexpected outcome. Therefore, we adopt the multiplier preference utility model proposed by Hansen and Sargent (2001) to capture decision makers' aversion to model uncertainty. Through modeling the catastrophe reinsurance market via a static Cournot competition model, we find that reinsurers’ aversion to model uncertainty will induce a cost effect which is determined both by the accuracy, i.e., the variance/variability, of the estimator and by the level of reinsurer’s aversion to model uncertainty. And such cost effect would result in a higher price of the coverage and a lower level of the supply. Moreover, such cost effect could be more pronounced under the situation where reinsurers underwrite risks that are subject to positively correlated model uncertainties, because the additional covariance has to be taken into consideration as well.

Heavy-tailed probability distribution plays a key role in modeling catastrophe risks. A well known fact is that diversifying heavy-tail distributed risks may not always be optimal and can result in the so called “nondiversification” trap in catastrophe insurance market (Imbragimov et al. 2009). Among the commonly used heavy-tailed distribution, the generalized Pareto distribution receives particular interest, because it is the limiting distribution of excess loss. In this paper, we draw the connection between generalized Pareto distribution and decision maker’s aversion to model uncertainty and examine the diversification effect of catastrophe risks from an entropic point of view.

The reminder of this paper is organized as follows. In section 2, we briefly review the ambiguity aversion preferences with an emphasis on the multiplier utility preference proposed by Hansen and Sargent (2001). We discuss the characteristics and the current stage of the catastrophe insurance market. In section 3, we apply the multiplier utility preference to address
the issue of limited participation in the catastrophe reinsurance market due to catastrophe model uncertainty. In section 4, we show that generalized Pareto distribution can be obtained via decision maker’s entropy optimization under certain proper constraints. We discuss the properties of the generalized Pareto distribution with respect to decision maker’s aversion to model uncertainty. In section 5, we discuss limitations of the multiplier utility preference model and draw the conclusion.

2. Related Literature

2.1. Model Uncertainty and Ambiguity Aversion

Under the classic subjective expect utility (SEU) preference, the decision maker ranks the random payoff \( h \) according to

\[
V(h) = \max_u \int u(h)dP \quad (2.1)
\]

where \( u \) denotes the standard utility function and \( P \) is the subjective probability distribution that describes the states of the world. Under the SEU preference, the probability distribution \( P \) is unique, i.e., the decision maker has full confidence in such a probability distribution.

Generally speaking, risk refers to the randomness with known probability distribution and uncertainty refers to the situation where probability distribution is unknown. Thought experiments conducted by Ellsberg (1961) and a bunch of other researchers (e.g., Harlevy 2007) reveal that decision makers are in general not neutral between risk and uncertainty. Such preference cannot be rationalized under the classic SEU preference framework. The strand of literature in ambiguity and ambiguity aversion thus deviate from the SEU preference in that decision makers do not fully trust the underlying/prior probability distribution due to lack of complete information or knowledge, and instead have a set of prior probability distributions that
may be plausible. Various preference models associated with ambiguity aversion have been proposed, including the multiple-prior model (Gilboa and Schmeidler, 1989), the smooth ambiguity model (Klibanoff et al., 2005), the multiplier utility preference model and its related robust control theory (Hansen and Sargent 2001, 2006).

The multiple-prior model proposed by Gilboa and Schmeidler (1989) is also known as the max-min expected utility model. The decision maker is assumed to choose the minimum (the worst-case scenario) of the maximized utilities over a set of possible prior probability distributions. This approach has been criticized because it assumes that the decision maker holds an extreme pessimistic attitude. To circumvent this flaw, Ghirardato et al. (2004) propose a generalization of this preference structure known as the $\alpha$ max-min expected utility model, under which the decision maker’s objective function is assumed to be a convex combination of the minimum (“pessimistic”) and maximum (“optimistic”) over a set of expected utilities with the assigned weight of $\alpha$ and $1-\alpha$ respectively. When $\alpha=1$, it reduces to the max-min expected utility, and when $\alpha=1/2$, the decision maker is assumed to be ambiguity neutral.

Klibanoff et al. (2005) propose a more complicated smooth ambiguity aversion preference in which the decision maker’s objective function is characterized by the triple $(\phi, u, \mu)$ through the following functional form.

$$V(h) = \max_{\phi, u} \int_{\Delta} \phi \left( \int_{\Omega} u(h(\omega)) dp(\omega) \right) d \mu(p) \quad (2.2)$$

Here $u$ is a standard increasing concave utility function. The information set $\mu \in \Delta$ contains a set of priors of the probability distribution $p \in \Omega$, if the decision maker were not certain about the true probability distribution. The distortion function $\phi$ is introduced to capture the decision maker’s attitude toward ambiguity. It functions like the standard utility function $u$. 
to capture the decision maker’s risk aversion. When \( \phi \) is an increasing concave function, the decision maker exhibits ambiguity aversion. The degree of ambiguity aversion increases as the function \( \phi \) becomes more concave. Under this paradigm, the decision maker’s objective can be viewed as a nested von Neumann-Morgenstern expected utility, e.g., “expectation over expectations”. The main advantage of this approach is that it provides the separation between ambiguity aversion and risk aversion. However, the complicated functional form makes it hard to derive explicit analytical results.

Motivated by the robust control theory, Hansen and Sargent (2001) propose the multiplier utility preference to model the decision maker’s robust decision making toward model uncertainty. For a fixed reference probability distribution \( P \in \Omega \), and for any other probability distribution \( Q \in \Omega \), define the relative entropy as

\[
I(Q \| P) = \int_{\Omega} \log \left( \frac{dQ}{dP} \right) dQ
\] (2.3)

The decision maker’s objective function can be formulated as a multiplier utility model (Hansen and Sargent, 2006) as follows.

\[
V(h) = \max_u \min_{\phi, \eta} \int_{\Omega} u(h) dQ + \theta \left[ I(Q \| P) - \eta \right] (\theta > 0)
\] (2.4)

The above multiplier utility preference structure can be interpreted as follows. The decision maker has some best “guess” of the probability distribution \( P \) but does not have full confidence with it. Instead, he considers a set of alternative probability distributions that may be plausible. However, since the reference model \( P \) is the best “guess” based on available information, the decision maker will penalize his deviation from the reference probability \( P \) to an alternative probability \( Q \) by the relative entropy. The relative entropy, also known as the Kullback–Leibler
divergence, can be viewed as a measure of “distance” between two probability distributions. The relative entropy is non-negative, convex, and equals to 0 only if \( P = Q \).

Matched with the imposed relative entropy constraint \( \eta \), the parameter \( \theta \) captures the degree of trust of the decision maker in the reference model \( P \), or conversely, the aversion to the misspecification of the reference model. A higher value of \( \theta \) reflects a higher degree of trust in the reference model, or in other words, a lower degree of aversion to model misspecification. In the limit case, i.e., \( \theta \to \infty \), the decision maker fully trusts the reference probability distribution \( P \) or shows no aversion to model uncertainty. When \( \theta < \infty \), the decision maker shows some degree of aversion to model uncertainty and will make his decision based on the worst-case misspecification probability distribution \( Q \), which is the solution to the relative entropy minimization problem. His degree of aversion towards model uncertainty can go to infinity as \( \theta \) approaches 0.

Practically, the optimization problem under the multiplier utility preference in equation (2.4) can be solved in two steps: (1) the decision maker first solves the inner relative entropy minimization problem to find the worst-case misspecification probability distribution; (2) he then maximizes his expected utility based on the worst-case misspecification probability distribution.

We are motivated to adopt the multiplier utility preference structure in this paper for several reasons. Firstly, the multiplier utility preference provides us with a general framework to model insurers’ decision making under incomplete/imprecise information and related consequences to the insurance market. This approach builds on more meaningful behavioral assumptions regarding the decision maker’s response to model uncertainty. Secondly, the relative entropy minimization criterion under the multiplier utility preference can also be connected to the probability distortion that is commonly used in insurance literature. Reesor and MacLeish
(2002) provide an extensive treatment between the relative entropy minimization and probability distortion. They show that for exponential family of distributions, the relative entropy minimization problem is equivalent to the distortion of the original probability distribution under certain moment constraints. Moreover, the distorted probability distribution will remain as the same distribution but with changed canonical parameters. This provides us with the mathematical tractability for a large family of heavily used distributions in finance and insurance, such as normal distribution and Poisson distribution. Actually, it can be shown that the worst-case misspecification probability distribution $Q$, e.g., the solution to the relative entropy minimization problem, is equivalent to apply the Esscher transform to the reference probability distribution $P$ (see Pham, 2010, proposition 2.1). Denote $f_0$ and $f_1$ the reference probability distribution and the worse-case misspecification probability distribution, respectively. $f_1$ can be expressed as (see appendix A)

$$f_1(h) = f_0(h) \frac{\exp\left(-\frac{u(h)}{\theta}\right)}{E_{f_0}\left[\exp\left(-\frac{u(h)}{\theta}\right)\right]} \quad (2.5)$$

The Esscher transform has a long history in insurance literature. For instance, Buhlmann (1984) shows that the Esscher transform can be obtained as equilibrium pricing in a risk-sharing economy. The multiplier utility preference may thus provide alternative interpretations of this classical result in insurance literature. Less mentioned in the economics literature, the probability distortion under the multiplier utility preference may also be justified by the large deviation theory and importance sampling, in which the Laplace principle (Esscher transform) and relative entropy minimization play important roles in deriving asymptotic results on rare event probabilities (Pham, 2010).
2.2. Catastrophe Insurance Market

Several anomalies of the catastrophe (re)insurance market have been documented in insurance literature (see Froot, 2001 for a complete review), including (1) the coverage for catastrophe risks are often limited and can become unavailable after the occurrence of a major catastrophe event; (2) the pricing of the coverage is not actuarially fair and can be multiples of the expected loss; (3) the supply of the coverage and the pricing of the coverage are subject to cycles with alternate soft and hard markets.

The distinct features of catastrophe risks, along with various inefficiencies that can affect either the supply or the demand for catastrophe insurance, may help to explain the disruptions in catastrophe (re)insurance market.

(1) Distinct statistical properties of catastrophe risks compared with other insurance risks.

(a) The size of catastrophe risks can be too big to be classified as insurable or diversifiable risk (Cummins, 2007). Moreover, there may exist a positive correlation between catastrophe risks and other risks in insurers' portfolio. This feature can violate the assumption of the Law of Large Numbers and thus break down the traditional risk warehouse model (Cummins and Trainar, 2009).

(b) Catastrophe risks are usually heavy-tailed, meaning larger loss variability and skewed distribution. Imbragimov et al. (2009) study limited participation in the catastrophe insurance market, termed as “nondiversification trap”. They show that when insurers with limited liability face constraints on both the aggregate amount of risks and the size of an individual risk that they can take, adding heavy-tailed catastrophe risks to their insurance portfolios can be inferior. The nondiversification trap, in which no insurance will be offered, can arise when the total number of participants fails to
reach a critical level, even though there is enough capacity to attain full risk sharing. They therefore view limited participation in the catastrophe insurance market as a coordination problem. Froot (2007) shows that the compensation for the skewness of the distribution is an important pricing factor to the (re)insurer’s charged hurdle rate. Therefore, the inherent skewness related to the loss distribution of catastrophe risks may explain the high price for the coverage observed in the catastrophe insurance market.

(c) Because of the limited historical data of catastrophe events, the estimation of catastrophe risk distributions may not be accurate or stable over time. Changes in the price and coverage in the post-event catastrophe insurance market can be caused by (re)insurers’ probability updating (Cummins and Lewis, 2003).

(2) Limited capital/Capacity constraint. Because the direct effect of unexpected losses caused by catastrophe events is the depletion of insurers’ or even entire insurance industry’s capital, the literature in underwriting cycle (Winter, 1994; Cummins and Danzon, 1997; Weiss and Chung, 2004) can shed light on the disruptions in the post-event catastrophe insurance market. That is, the high price and limited coverage in the post-event catastrophe insurance market can be explained by shortage of capital.

(3) Higher frictional costs. Frictional costs can arise due to customers’ aversion to insurers’ insolvency (Cummins and Danzon, 1997; Zanjani, 2002; and Froot, 2007), high transaction costs resulting from illiquid assets or hedging tools such as reinsurance contracts (Froot, 2001) or counter-party risk that compromises the long-term insurance-reinsurance relationship (Cummins and Lewis, 2003), the agency problem associated with the corporate firm (Froot and O’Connell, 1999; Froot, 2001; and Froot, 2007).
Higher frictional costs can lead to higher costs of capital that restricts the supply in the catastrophe insurance market.

(4) Market power of reinsurers. Reinsurers’ market power intensifies over time with an increase in capital and market share (Cummins and Weiss, 2000). The increasing market power of reinsurers may lead to less efficient market equilibrium with a higher price and a lower supply of the coverage, compared with those under full competition.

(5) Behaviors of rating agency. Froot (2008) argues that to achieve good diversification of risk, rating agencies encourage (re)insurers to spread their capital across all risk areas. But in doing so, they provide relatively too much capital to these diversifying exposures, which are smaller in size, and relatively too little capital to the larger U.S. perils. The result is that inadequate capital is left for the U.S. market where the need is actually greatest.

(6) Regulation, accounting and taxation factors. Rate suppression, rate compression, unfavorable regulatory accounting treatment introduced by the regulation can distort the market price and create the adverse selection problem in the catastrophe insurance market (Cummins, 2007). Particular to catastrophe risks, the existing program to mitigate the catastrophe risks and the ex-post intervention by third-parties can reduce the demand for insurance and cause the moral hazard problem (Froot, 2001; Cummins, 2007).

Some of the above mentioned phenomena in the catastrophe insurance market, such as relatively high market prices of CAT reinsurance and CAT bonds, are hard to be reconciled by taking into account uncertainties associated with catastrophe models under the classic expected utility framework. Froot (2001) holds the view that under the expected utility framework, parameter uncertainty associated with catastrophe models cannot justify the pricing anomaly in
the reinsurance market as long as the estimators for expected losses are unbiased. And learning, i.e., Bayesian style probability updating, could be very difficult in this market due to the rareness of catastrophe events. Froot and Posner (2002) further demonstrate that under the expected utility framework, parameter uncertainty will not affect CAT bond pricing, because the unbiased disturbance term in the estimator of the occurrence of catastrophe events will not affect the moments of such risky assets. However, we would argue that it would be quite hard for any rational decision maker to ignore model uncertainty, especially when the level of uncertainty is so high that could even make the unbiased estimator uninformative. As we will show in following sections, introducing the decision maker’s aversion to model uncertainty may help to provide alternative explanations to those market disruptions.

3. Aversion to Model Uncertainty and its Implications to Catastrophe Insurance Market

In this section, we apply the multiplier utility preference to the catastrophe insurance market. Through a static Cournot competition model, we show that limited participation in the catastrophe reinsurance market can be justified by the cost effect induced by reinsurers’ aversion to catastrophe model uncertainty. We adopt the static Cournot competition model following Froot and O’Connell (1999, 2008) by recognizing the market power of reinsurers. We first consider the case of the stand-alone model uncertainty and then discuss the case for two correlated model uncertainties.

3.1. Catastrophe Risk Subject to Stand-alone Model Uncertainty

The basic model set up is as follows:

• There are N reinsurers in the catastrophe reinsurance market.

• Assume a linear inverse demand function in this market, e.g.,
\[ P(Q) = a - Q = a - \sum_{i=1}^{N} q_i \]

- Each reinsurer has a linear cost function and the marginal cost per unit coverage is assumed to be constant denoted by \( \bar{C} \). This marginal cost can be viewed as the estimated loss per unit coverage generated by the catastrophe model.

- Due to incomplete knowledge about the true exposure to catastrophe risks, reinsurers may suspect that the true marginal cost may be actually modeled by

\[ MC = \bar{C} + \varepsilon, \quad \varepsilon \sim N(0, \sigma^2) \]

where \( \varepsilon \) is a disturbance term that captures model uncertainty. Particularly, we assume that \( E(\varepsilon) = 0 \), e.g., the current marginal cost \( \bar{C} \) is an unbiased estimator of the true marginal cost.

- Reinsurers are assumed to be risk neutral but averse to model uncertainty. Reinsurers are assumed to maximize their profits. In order to account for catastrophe model uncertainty, reinsurers adopt the multiplier preference structure and solve the following optimization problem (Hansen and Sargent, 2006)

\[
\max_{q_i} \min_{f_i} \left( P(Q)q_i - q_i \int (\bar{C} + \varepsilon)f_i(\varepsilon)d\varepsilon - \theta \left[ \int \ln \left( \frac{f_i(\varepsilon)}{f_0(\varepsilon)} \right) f_i(\varepsilon) - \eta \right] \right) \tag{3.1}
\]

where \( f_0(\varepsilon) \) and \( f_i(\varepsilon) \) denote the reference probability distribution and the worst-case misspecification probability distribution respectively, \( \eta \) is the constraint on relative entropy imposed by reinsurers.

In order to solve the above problem, we first find out the worst-case misspecification probability distribution, which turns out to be a normal distribution with mean \( \sigma^2 / \theta \) and variance \( \sigma^2 \) (see appendix B). Next, we express the expected marginal cost under the worst-case misspecification probability distribution as a function of theta. We then can solve for the Nash
equilibrium market price ($P^{NE}$) and quantity ($Q^{NE}$) as in the standard Cournot competition game. We present the results in Table 2 and compare them with those when reinsurers show no aversion to model uncertainty.

**Table 1: Market Equilibrium Comparison**

<table>
<thead>
<tr>
<th></th>
<th>$Q^{NE}$</th>
<th>$P^{NE}$</th>
<th>$q_i^{NE}$</th>
<th>Marginal Cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>No Model Uncertainty</td>
<td>$\frac{N(a - \bar{C})}{N + 1}$</td>
<td>$\frac{a + NC}{N + 1}$</td>
<td>$\frac{a - \bar{C}}{(N + 1)}$</td>
<td>$\bar{C}$</td>
</tr>
<tr>
<td>Under Model Uncertainty</td>
<td>$\frac{N\left(a - \bar{C} - \frac{\sigma^2}{\theta}\right)}{N + 1}$</td>
<td>$\frac{a + N\left(\bar{C} + \frac{\sigma^2}{\theta}\right)}{N + 1}$</td>
<td>$\frac{a - \left(\bar{C} + \frac{\sigma^2}{\theta}\right)}{(N + 1)}$</td>
<td>$\bar{C} + \frac{\sigma^2}{\theta}$</td>
</tr>
</tbody>
</table>

At the firm level, when the reinsurer shows aversion to model uncertainty, its marginal cost will be higher under the worst-case misspecification distribution than under the reference distribution. The extra loading per unit coverage, which is analogous to the variance premium principle, depends on (1) the precision of the estimator of the marginal cost $\sigma^2 = \text{Var}(MC)$; (2) the level of confidence in the model, or conversely, the level of aversion to model uncertainty, captured by $\theta$. For a fixed $\sigma^2$, when the reinsurer fully trusts the model, i.e., $\theta \to \infty$, the extra loading becomes zero. On the other hand, as the level of the reinsurer’s aversion to model uncertainty increases, i.e., $\theta \to 0$, the extra loading increases accordingly, leading to a lower level of the supply of coverage in the market. Clearly, the reinsurer will not offer any coverage when $a - \bar{C} - \frac{\sigma^2}{\theta} = 0$. That is, the degree of aversion to model uncertainty is so high that drives the expected profit to zero.
At the market level, when reinsurers exhibit aversion to model uncertainty, the market equilibrium quantity (price) is lower (higher) than those when they do not show aversion to model uncertainty. The whole catastrophe reinsurance market can break down when the level of aversion to model uncertainty is high enough. If we further relax the symmetric assumption that reinsurers have the same marginal cost and relative entropy constraint, this model will predict that reinsurers with higher levels of aversion to model uncertainty and/or higher marginal costs (possibly due to lower level of capital) will restrict their supply and even withdraw from the catastrophe reinsurance market. In short, reinsurers’ aversion to model uncertainty can result in a cost effect that leads to a lower level of supply and a higher price of coverage than normally would be. When reinsurers’ aversion to model uncertainty is higher than some threshold, limited participation or even exit from the market could occur.

3.2. Catastrophe Risks Subject to Correlated Model Uncertainty

We only consider the stand-alone catastrophe risk in case 1. Next, we explore the impact of reinsurers’ aversion to model uncertainties that may be correlated. For instance, the reinsurer may write the catastrophe reinsurance contracts regarding the losses caused by hurricanes within two adjacent costal states. The model set up is as follows.

- Here we assume that there are two reinsurance markets with a linear demand function as

$$P_i(Q_i) = a_i - b_iQ_i$$ \(i = 1, 2\).

- The marginal costs for each reinsurer \(i\) in market 1 and 2 are \(\bar{C}_1, \bar{C}_2\) respectively.

- Similar to case 1, the reinsurer suspects that the above marginal costs (possibly yielded by the catastrophe model) are only approximates, and that true marginal costs may be given by

$$MC_1 = \bar{C}_1 + \epsilon_1, MC_2 = \bar{C}_2 + \epsilon_2$$

where the random disturbance terms are assumed to follow a bivariate normal distribution as
\[ [\varepsilon_1, \varepsilon_2] \sim N(0, \Sigma), \Sigma = \begin{pmatrix} \sigma_1^2 & \rho_{12} \sigma_1 \sigma_2 \\ \rho_{12} \sigma_1 \sigma_2 & \sigma_2^2 \end{pmatrix} (\rho > 0) \]

The assumption regarding the joint distribution is similar to case 1 in that the current marginal costs are unbiased estimators. Reinsurers are assumed to maximize their profit by jointly determine the supplies in those two markets as follows.

\[
\max_{q_1, q_2} \min_{(q_1, q_2)} \sum_{i=1}^{2} \left\{ P_i q_i - q_i \left[ \int (C_i + \varepsilon_i) dF_i, \theta_i \left[ \int \ln \left( \frac{dF_i(\varepsilon_1, \varepsilon_2)}{dF_0(\varepsilon_1, \varepsilon_2)} \right) dF_i(\varepsilon_1, \varepsilon_2) - \eta_i \right] \right] \right\} \quad (3.2)
\]

In appendix C, we show that the marginal distribution of the disturbance terms still follow the normal distribution but with distorted mean that depends not only on the precision of the estimated loss and the level of aversion to model uncertainty in market 1, but also on (1) the covariance between the two random disturbance term \( \varepsilon_1 \) and \( \varepsilon_2 \); (2) the level of aversion to model uncertainty in market 2. The expected marginal cost in market 1 and market 2 under the worse-misspecification probability distribution are given by

\[
E_{F_i}(MC_1) = \bar{C}_1 + \frac{\sigma_1^2}{\theta_1} + \frac{\rho_{12} \sigma_1 \sigma_2}{\theta_2}
\]

\[
E_{F_i}(MC_2) = \bar{C}_2 + \frac{\sigma_2^2}{\theta_2} + \frac{\rho_{12} \sigma_1 \sigma_2}{\theta_1}
\]

The above formula can be viewed as an analog to the covariance premium principle. More importantly, when \( \rho_{12} > 0 \), it is easy to see that the marginal cost in each market will be bigger than that of the stand-alone case. One direct consequence is that the price (quantity) for the catastrophe reinsurance in each market will further increase (decrease). In short, with the existence of positively correlated model uncertainties, reinsurer’s aversion to model uncertainty can make the disequilibria in the catastrophe reinsurance market in terms of price and quantity more severe.
3.3. Summary

In above examples, we implicitly assume that reinsurers are risk neutral but averse to the model uncertainty because of the adopted linear profit function, which allows us to derive the closed form solutions. The results indicate that the reinsurer's aversion to model uncertainty under the multiplier utility preference will lead to a cost effect, even if the current estimator of the marginal cost is an unbiased estimator. Moreover, the cost effect can be more pronounced when reinsurers are facing risks that are subject to correlated model uncertainties. One direct consequence is that the catastrophe reinsurance market will become even less efficient than the one with the standard oligopoly market structure.

The above simple model may also help to explain several observed disequilibria in the catastrophe insurance market. Firstly, it may help to explain the relative higher price of the coverage. Considering that the variability of the estimator yielded by the catastrophe models can be multiples of the estimator (Guy Carpenter, 2011), such imprecise information will lead to a high equilibrium price that can be multiples of the expected loss, which is consistent with the empirical evidence (Froot, 2001). Secondly, because the localized insurers face a higher degree of imprecise information of potential exposure to catastrophe risks, the model indicates that localized small insurers will not participate in the market if they exhibit high levels of aversion to model uncertainty. This may provide alternative explanation for limited participation in the catastrophe insurance market. What we try to emphasize is that, different from the probability updating in the post-event period, imprecise information could be a persistent problem that creates additional frictional costs in efficiently sharing catastrophe risks. Lastly, this model also implies that transferring the insurance liability that is subject to model uncertainty could help reinsurers to mitigate such cost effects by creating a stand-alone entity that bears both risk and
uncertainty. On the flip side of the cost effect induced by reinsurers’ aversion to model uncertainty is that the capital requirement for the catastrophe insurance will be higher. Given the total fixed capital level, this means that catastrophe risk will “consume” more capital. Therefore, as long as the reinsurers’ premium paid to CAT bonds is less than the received premium through reinsurance contract, the issue of CAT bonds will be a positive NPV project to the reinsurer with additional benefits as such transactions will not impose extra capital requirement to reinsurers.

4. Stable distribution for Excess Loss Variable under Entropy Optimization

In statistical inference literature, two entropy-based principles are of particular interest. The first is called the principle of maximum entropy developed by Jaynes (1957, 2003). It states that given partial information or constraints about an unknown probability distribution, the probability distribution that is the least biased is the one with the maximum entropy. When a prior distribution is specified, the above principle can be extended to the principle of minimum cross entropy, which states that among all probability distributions that satisfy a given set of constraints, the one with minimum relative entropy with respect to the prior distribution is the one that is maximally committed to the prior. From the perspective of entropy optimization, every probability distribution can be obtained by either maximizing an appropriate entropy measure or minimizing a cross-entropy measure with respect to an appropriate prior distribution, subject to appropriate constraints (Kapur and Kesavan, 1992, p.297). In this section, we show that the generalized Pareto distribution (GPD) can be obtained through entropy optimization. This has important implications to catastrophe reinsurance markets because reinsurers have to bear excess loss above than some threshold level and GPD is the limiting distribution of the excess loss random variable.
Proposition 4.1: Given the relative entropy constraint between the prior and posterior probability distributions and an additional mean constraint of the posterior distribution, the right tail of the posterior distribution that satisfies those constraints follows a generalized Pareto distribution.

The entropy optimization problem in Proposition 4.1 can be written as follows.

\[
\max_Q H(Q) = -\int Q(x) \log Q(x) dx \\
\text{s.t., } I(Q \parallel P) = \int Q(x) \log \left( \frac{Q(x)}{P(x)} \right) dx = \eta \quad (4.1)
\]

\[
m = E_q(X) = \int xQ(x) dx
\]

where \( P(x) \) and \( Q(x) \) are the prior and posterior probability density functions and the objective function \( H(Q) \) is defined as the classical Shannon entropy. Proposition 4.1 states that the right tail of the optimal solution, \( Q(x) \), to the above problem follows a generalized Pareto distribution with shape parameter \((\theta - 1) > 0\) and scale parameter \(\beta > 0\), i.e.,

\[
Q(x) \propto \left[ 1 + \frac{\beta}{\theta - 1} (x - m) \right]^{-(\theta - 1) - 1} \quad (4.2)
\]

where the parameter \(\theta, \beta\) are the Lagrange parameters such that the relative entropy constraint and the first moment constraint hold respectively.

Several properties of the tail behavior of the optimal distribution can be discussed here. Firstly, the shape parameter is determined by the Lagrange parameter associated with the relative entropy constraint. If we apply the same logic for the multiplier preference discussed in Section 3 to interpret this shape parameter, we can conclude that a higher level of model uncertainty aversion (i.e., a smaller \(\theta\)) leads to a more risky right tail distribution, and vice versa. When \(\theta \to +\infty\), we have an exponentially decayed right tail.
Secondly, the tail distribution is stable with respect to different thresholds, leaving the shape parameter unchanged. For instance, if \( d \) is the threshold, the density function of the excess loss can be shown as

\[
Q_d(x) = Q(x+d) \propto \left[ 1 + \frac{\beta}{\theta-1} (x+d-m) \right]^{-(\theta-1)-1} \propto \left[ 1 + \frac{\beta'}{\theta-1} (x-m) \right]^{-(\theta-1)-1}
\]

where \( \beta' = \beta / (1 + \beta d) \). That is to say, the tail index of the right tail distribution, i.e., the riskiness, is unchanged, and only the scale parameter changes.

With those properties in mind, we immediately draw some conclusions between aversion to model uncertainty and the related literature that study the heavy-tailed distributions and the diversification effect (see, for example, Imbragimov et al. 2009 and Zhou 2010). Specifically, when the level of aversion to model uncertainty is high such that the tail index \((\theta-1) < 1 \Rightarrow \theta < 2\), under the Value-at-Risk measure there is a negative diversification effect (see Zhou 2010, Theorem 4.1). That is, aggregating the heavy-tailed risks with the tail index less than 1 results in a more risky portfolio in terms of VaR.

5. Conclusions

We argue that incomplete knowledge and imprecise information could be a persistent problem and represent a significant source of frictional costs in catastrophe reinsurance markets, where market participants are constantly fighting for the unexpected. Thus, by allowing decision makers to deviate from the standard expected utility framework, we show that aversion to model uncertainty can create additional frictional costs and cause market disruptions in catastrophe reinsurance markets.
In the literature related to ambiguity aversion, the multiplier utility preference has been criticized since it is observationally indistinguishable from the subjected expected utility (Epstein, 2010), because introducing the parameter $\theta$ is indistinguishable from the increasing level of risk aversion. However, we believe that the decision criterion based on the relative entropy measure, along with the Esscher transform of probability distributions, under the multiplier utility preference may help to interpret some of the stylized results such as the variance and covariance premium principles and provide new insights to the disruptions in catastrophe reinsurance markets. Possible extensions to this paper may include extending the static Cournot competition game into a dynamic model to address the possible effects of aversion to model uncertainty on reinsurers’ capital decision. It would also be interesting to study the relationship between insurers and reinsurers under asymmetric information using the ambiguity aversion framework. For instance, does there exist an optimal risk sharing structure between insurers and reinsurers under model uncertainty?
References


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**Appendix**

**A. Proof of the Worst-Case Misspecification Probability under the Multiplier Preference**

Denote $f_0, f_1$ the reference probability distribution and the worst-case misspecification probability distribution respectively. Given the multiplier preference defined in equation (2.5), we have

$$V(h) = \max_u \min_{\theta} \int u(h)f_1(h)dh + \theta \left[ \int \ln \left( \frac{f_1(h)}{f_0(h)} \right) f_1(h)dh - \eta \right]$$

In order to find the worse-case misspecification probability distribution, we focus on the inner relative entropy minimization problem. Firstly, we shall note that relative entropy is nonnegative and equals to 0 only if $f_1 = f_0$. Therefore, to prevent a degenerated solution of the relative entropy minimization problem, we need $\theta > 0$ and the value of $\theta$ is determined by the imposed constraint on the relative entropy $\eta$.

The first order condition with respect to $f_1$ is given by

$$u(h) + \theta \left[ \ln f_1(h) - \ln f_0(h) + 1 \right] + k = 0$$

where $k$ is a constant introduced by the condition that $\int f_1(h)dh = 1$.

Rearrange the terms, we have

$$f_1(h) = c \exp \left( -\frac{u(h)}{\theta} \right) f_0(h)$$

where $c$ is a normalizing constant.
Given the condition that \( \int f_1(h) dh = 1 \) and the assumption that \( \int \exp\left(-\frac{u(h)}{\theta}\right) f_0(h) dh < \infty \),

the normalizing constant can be easily identified as

\[
c = \frac{1}{\int \exp\left(-\frac{u(h)}{\theta}\right) f_0(h) dh} = \frac{1}{E_{f_0}\left[\exp\left(-\frac{u(h)}{\theta}\right)\right]}
\]

Therefore, the worst-case misspecification probability density function under the multiplier preference can be represented as

\[
f_1(h) = \frac{\exp\left(-\frac{u(h)}{\theta}\right)}{E_{f_0}\left[\exp\left(-\frac{u(h)}{\theta}\right)\right]} f_0(h)
\]

### B. Proof of case 1-Stand-Alone Model Uncertainty

Following Hansen and Sargent (2008), we first need to solve the inner relative entropy minimization problem defined in equation (3.1) to find the worst-case misspecification probability distribution of \( \varepsilon \).

Denote \( f_0, f_1 \) the reference and worst-case misspecification probability distribution respectively. The first order condition for the relative entropy minimization is given by

\[
\theta \left[ \ln f_1(\varepsilon) - \ln f_0(\varepsilon) + 1 \right] + k = (\bar{C} + \varepsilon)
\]

where \( k \) is a normalizing constant introduced by the condition that \( \int f_1 = 1 \).

We can find that the worst-case misspecification probability distribution is proportional to the reference probability distribution as

\[
f_1(\varepsilon) \propto \exp\left(\frac{\varepsilon}{\theta}\right) f_0(\varepsilon) = \exp\left(\frac{\varepsilon}{\theta}\right) \exp\left(-\frac{\varepsilon^2}{2\sigma^2}\right)
\]
We need to choose a normalizing constant such that the density \( f_\varepsilon(\varepsilon) \) integrates to unity. And such constant exists only when the following condition holds.

\[
\int \exp\left(\frac{\varepsilon}{\theta}\right) \exp\left(-\frac{\varepsilon^2}{2\sigma^2}\right) d\varepsilon < \infty \quad (B.3)
\]

The integral in equation (A.3) is finite when \( \theta > 0 \). And this condition also guarantees that the relative entropy constraint is binding because the convexity of relative entropy.

It is straightforward to see that the density \( f_\varepsilon(\varepsilon) \) is also a normal distribution. We can find its mean and variance by completing the squares in the exponential terms as follow.

\[
\frac{\varepsilon}{\theta} - \frac{\varepsilon^2}{2\sigma^2} = -\frac{\theta\varepsilon^2 - 2\sigma^2\varepsilon}{2\sigma^2\theta} = -\frac{(\varepsilon - \sigma^2/\theta)^2}{2\sigma^2} + c \quad (B.4)
\]

where \( c \) is a constant that does not depend on \( \varepsilon \). This shows clearly that \( f_\varepsilon(\varepsilon) \) has a normal distribution with mean \( \sigma^2/\theta \) and variance \( \sigma^2 \).

Next, we can use \( f_\varepsilon(\varepsilon) \) to find the marginal cost based on the worst-case specification probability distribution as follow.

\[
\int (\bar{C} + \varepsilon) f_\varepsilon(\varepsilon) d\varepsilon = \bar{C} + \int \varepsilon f_\varepsilon(\varepsilon) d\varepsilon = \bar{C} + E_{f_\varepsilon}[\varepsilon] = \bar{C} + \frac{\sigma^2}{\theta} \quad (B.5)
\]

After solving the inner minimization problem, we can proceed to solve the standard Cournot competition problem. Reinsurer \( i \)'s best response function determined by the first order condition is given by

\[
FOC(q_i): P'(Q)q_i + P(Q) - \left(\bar{C} + \frac{\sigma^2}{\theta}\right) = 0
\]

\[
\rightarrow -q_i + P(Q) - \left(\bar{C} + \frac{\sigma^2}{\theta}\right) = 0
\]

Sum up the FOCs for \( N \) reinsurers, we can solve for the equilibrium price, \( P^{NE} \), as follow.
\[-\sum_{i=1}^{N} q_i + NP^{NE} - N\left(\bar{C} + \frac{\sigma^2}{\theta}\right) = 0\]

\[\rightarrow P^{NE} = a + NP(Q) - N\left(\bar{C} + \frac{\sigma^2}{\theta}\right) = 0 \quad (B.6)\]

\[\rightarrow P^{NE} = \frac{a + N\left(\bar{C} + \frac{\sigma^2}{\theta}\right)}{N + 1}\]

Substituting the equilibrium price back to the demand function and each reinsurer's best response function, we can solve the market equilibrium quantity and each reinsurer's equilibrium quantity as

\[q_i^{NE} = \frac{a - \left(\bar{C} + \frac{\sigma^2}{\theta}\right)}{(N + 1)} \quad (B.7)\]

\[Q^{NE} = \frac{N\left(a - \bar{C} - \frac{\sigma^2}{\theta}\right)}{N + 1} \quad (B.8)\]

C. Proof of case 2-Correlated Model Uncertainties

We first prove the general case for the correlated random disturbance terms that follow the joint normal distribution in M reinsurance markets.

Denote \(\varepsilon = [\varepsilon_1, \ldots, \varepsilon_M]\) as the vector for the random disturbance terms that follow joint normal distribution \(f_\varepsilon(\varepsilon) \sim N(0, \Sigma)\), where \(\theta = [0, \ldots, 0]_{i=M}\) and \(\Sigma = (\sigma_{ij})_{M \times M}\) is the covariance matrix. Let \(\theta = \left[\frac{1}{\theta_1}, \ldots, \frac{1}{\theta_M}\right] (\theta > 0, \forall i = 1, \ldots, M)\) denote the vector for the reinsurer’s level of aversion to model uncertainty for each reinsurance market.
The moment generating function for \( \varepsilon \) under the reference probability distribution is then given by

\[
M_{\varepsilon}(t) = \exp\left( \frac{t^T \Sigma t}{2} \right) \tag{C.1}
\]

Similar to proof in appendix B, the worst-misspecification distribution will be the Esscher transform of the reference probability distribution. The moment generating function under the worst-misspecification distribution with uncertainty parameter \( \theta \) becomes

\[
M_{\varepsilon}(t; \theta) = \frac{M_{\varepsilon}(t + \theta)}{M_{\varepsilon}(\theta)} = \frac{\exp\left( \frac{(t + \theta)^T \Sigma (t + \theta)}{2} \right)}{\exp\left( \frac{\theta^T \Sigma \theta}{2} \right)} = \exp\left( t^T \Sigma \theta + \frac{t^T \Sigma t}{2} \right) \tag{C.2}
\]

i.e., the Esscher transform (with parameter vector \( \theta \)) of a multivariate normal distribution is again a multivariate normal distribution, with mean \( \Sigma \theta \) and variance-covariance matrix \( \Sigma \).

Now, when we have two reinsurance markets, i.e. \( M=2 \), the expected mean for \( \varepsilon_1 \) in market 1 under the worst-case misspecification distribution will equal to \( E_{\varepsilon_1} = \frac{\sigma_1^2}{\theta_1} + \frac{\rho_{12} \sigma_1 \sigma_2}{\theta_2} \).

If we assume that \( \rho_{12} > 0 \), i.e. there is positive correlation between the two random disturbance term, we have \( E_{\varepsilon_1} = \frac{\sigma_1^2}{\theta_1} + \frac{\rho_{12} \sigma_1 \sigma_2}{\theta_2} > \frac{\sigma_1^2}{\theta_1} \). That is, the mean value of the expected loss under distorted distribution will be bigger than that of the stand-alone case in appendix B. And when \( \rho_{12} = 0 \), we can recover the result in appendix B.

**D. Proof of Proposition 4.1**
Following the mathematical treatment in Bercher (2008) and Bercher and Vignat (2008),
the proof of proposition 4.1 is as follow.

Denote $P(X), Q(X)$ the original and distorted probability density function respectively.

We first find the maximum entropy distribution $P(X)$ that satisfies the relative entropy constraints as follow.

$$\max_{Q} H(Q) = -\int Q(x) \log Q(x) dx$$

$$\text{s.t.} I(Q \parallel P) = \int Q(x) \log \left( \frac{Q(x)}{P(x)} \right) dx = \eta \quad (D.1)$$

where the objective function is the Shannon entropy measure.

The maximum entropy distribution in (D.1) can be shown as

$$Q(x) = \frac{1}{Z(\theta)} \exp \left( \theta \log \frac{Q(x)}{P(x)} \right) \quad (D.2)$$

where $\theta$ is the Lagrange parameter such that $I(Q \parallel P) = \eta$ and

$$\log Z(\theta) = (1 - \theta) \int P(x)^{\theta} dx \quad (D.3)$$

To simply the notation, define $q = \theta / (\theta - 1)$. Equation (D.2) and (D.3) together imply that the maximum entropy distribution can be expressed as

$$Q(x) = \frac{P(x)^q}{\int P(x)^q dx} \quad (D.4)$$

where we assume that $\int P(x)^q dx < \infty$.

Using equation (D.4), the optimal solution to problem (D.1) can be shown as

$$H(Q) = -\theta \eta + \log Z(\theta)$$

$$= -\frac{2 - q}{1 - q} \eta + \frac{1}{1 - q} \log \int P(x)^q dx \quad (D.5)$$

Now we introduce the additional mean constraint under the distorted probability distribution, i.e.,
\[ m = E_Q(X) = \int xQ(x)dx = \frac{1}{\int P(x)^{\theta} dx} \int xP(x)^{\theta} dx \quad (D.6) \]

Given this additional constraint, we need to further optimize the entropy based on (D.5) and have the following optimization problem

\[
\max_q \frac{1}{1-q} \log \int P(x)^{\theta} dx \\
\text{s.t. } \frac{1}{\int P(x)^{\theta} dx} \int xP(x)^{\theta} dx = m \quad (D.7)
\]

The solution to (D.7) is given by

\[ P(x) \propto [1 - \beta(1-q)(x-m)]^{\frac{1}{1-q}} \quad (D.8) \]

where the Lagrange parameter such that the mean constraint in (D.6) holds.

Using (D.4) and definition \( q = \theta / (\theta - 1) \), we have

\[
Q(x) \propto [1 - \beta(1-q)(x-m)]^{\frac{q}{q-1}} = [1 + \beta(q-1)(x-m)]^{-\frac{1}{q-1}} \\
= \left[ 1 + \frac{\beta}{\theta-1}(x-m) \right]^{-(\theta-1)^{-1}} \quad (D.9)
\]

Note that (D.9) is a generalized Pareto distribution with shape parameter \( (\theta - 1) > 0 \) and scale parameter \( \beta > 0 \).