Optimal Collateralization with Bilateral Default Risk

Daniel Bauer∗ Enrico Biffis† Luz Rocio Sotomayor†

March 13, 2013

PRELIMINARY AND INCOMPLETE

Abstract

We consider over-the-counter (OTC) transactions with bilateral default risk, and study the optimal design of the Credit Support Annex (CSA). In a setting where agents have access to a trading technology, default penalties and collateral costs arise endogenously as a result of foregone investment opportunities. We show how the optimal CSA trades off the costs of the collateralization procedure against the reduction in exposure to counterparty risk and expected default losses. The results are used to provide insights on the key drivers of different types of collateral rules, including hedging motives, re-hypothecation of collateral, and close-out conventions.

∗Bauer (dbauer@gsu.edu) and Sotomayor (rsotomayor@gsu.edu) are at the Department of Risk Management and Insurance, Georgia State University, 35 Broad Street, Atlanta, GA 30303, USA.
†Biffis (E.Biffis@imperial.ac.uk) is at Imperial College Business School, Imperial College London, South Kensington Campus, SW7 2AZ United Kingdom.
1 Introduction

The relevance of counterparty risk\(^1\) for the structuring and pricing of financial transactions has become apparent during the recent financial crisis. The master agreement of a derivative transaction, which a few years ago would have been mainly of interest to the legal department rather than the structuring desk of a bank, is now an integral part of the design and pricing of a deal. At a higher level, the mark-to-market gains or losses arising from counterparty risk\(^2\) have a significant impact on the earning results of financial institutions, and are therefore systematically monitored and hedged. Similarly, a variety of discount curves is now used for the valuation of assets and liabilities, to reflect the liquidity and counterparty risk profiles of different overnight indices and tenors.\(^3\) The new Dodd-Frank regulation in the US, and EMIR provisions in Europe, all emphasize the role of credit risk mitigation tools such as collateralization and segregation of collateral to limit the exposure of OTC markets to systemic risk. Despite the importance of these issues, little theoretical guidance is available to understand the design of the Credit Support Annex (CSA), to justify the form of specific collateral rules, and to quantify the costs and benefits associated with collateralization.

In this article, we consider two credit-risky counterparties that are homogeneous with respect to risk preferences, credit quality, and investment opportunities, but have opposite exposure to a random payoff that is nontradable and uspanned by the tradeable assets. This is a prototypical situation where agents can improve their position by sharing risk. There is a catch, however: the credit riskiness of the counterparties means that

\(^1\)Basel II (2006, Annex 4) defines counterparty risk as ‘the risk that the counterparty to a transaction could default before the final settlement of the transaction’s cash flows’.  
\(^2\)See Brigo et al. (2012b); Gregory (2012) for an extensive treatment of Credit/Debit Valuation Adjustment (CVA/DVA) and extensions.  
\(^3\)The interbank market now quotes very diverse credit and liquidity premia, even for plain vanilla products.
any risk sharing arrangement will be exposed to counterparty risk. Any counterparty risk mitigation tool (such as collateralization), on the other hand, will divert resources away from profitable trading opportunities, and hence give rise to an opportunity cost of posting collateral. Even if agents have symmetric collateral requirements, the costs of collateralization need to be weighted against the expected reduction in default penalties that collateralization is able to deliver. Moreover, the collateralization procedure will result in the illiquid exposure being regularly marked-to-market, therefore exposing the agents’ traded wealth to an additional, potentially unrewarded, source of risk. We develop a model that allows us to formalize these issues and quantify the main trade-offs at play, therefore providing tools that can be used to answer a number of interesting questions. First, we provide a rational for collateral rules commonly observed in OTC markets, by endogenizing the collateral rules as a result of the agents’ hedging demand against expected default penalties. Second, by endogenizing both the CSA and the cost of posting collateral, we show how funding costs and the CSA shape the valuation of OTC transactions. Third, we provide a robust framework to support the analysis of relevant policy issues, such as the potential impact of mandating suboptimal collateralization levels and moving to centralized clearing for OTC derivatives.

Our contribution is related to at least three different strands of literature. We contribute to the analysis of bilateral default risk, initiated in Rendleman (1992) and Duffie and Huang (1996), by explicitly allowing for collateral when defining default losses and valuing OTC instruments. Several contributions have recently looked at bilateral default risk\(^4\) and CSA pricing,\(^5\) but they either ignore collateralization or take collateral rules as exogenously given. Here, we explicitly look at the design of the CSA,

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\(^4\)See, for example, Brigo et al. (2012a); Crépey (2012a,b); Hull and White (2012).
\(^5\)See Biffis et al. (2011); Brigo et al. (2012c).
and obtain empirical predictions that shed light on counterparty risk mitigation tools commonly used in practice.

We also contribute to the discussion on how collateral costs enter the valuation of collateralized products. In the interest-rate swaps market, for example, it is well known that swaps valuation formulae based on par bond rates of a LIBOR-quality issuer (e.g., Duffie and Singleton, 1997) present a fundamental inconsistency, as interest rate swaps are typically fully collateralized (e.g., ISDA, 2010a) and hence counterparty risk should be negligible.\(^6\) Johannes and Sundaresan (2007) find evidence of costly collateral in interest rate swaps by comparing swap market data with swap values based on portfolios of futures and forward contracts, and by estimating a dynamic term structure model using Treasury and swap data. Brigo et al. (2012b) develop a framework to price OTC instruments in the presence of both collateralization and funding costs. As opposed to these studies, in our model collateral costs arise endogenously as the opportunity cost of diverting resources from trading opportunities. Even if collateral requirements are symmetric for the counterparties, we show how the cost of posting collateral enters the optimal design of the CSA and hence affects the valuation of collateralized transactions.

Finally, we offer relevant tools to quantify the potential impact of centralized clearing on the demand for collateral and trading volume\(^7\) in OTC derivatives markets (e.g., Singh and Aitken, 2009; Heller and Vause, 2012; Sidanis and Zikes, 2012). Our framework allows us to quantify the utility gains/losses to the counterparties when using different collateral rules, close-out conventions, and collateral treatment rules. Rather than abstracting from variation margins and reducing the costs of central clearing to the

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\(^6\)This is why econometric models of interest rate swap spreads may assume interest rate swaps to be free of credit risk (e.g., Collin-Dufresne and Solnik, 2001).

\(^7\)Another strand of literature studies the impact of central clearing on systemic risk in OTC markets (see Duffie and Zhu, 2011; Babus and Kondor, 2012, for example).
initial margin required by a clearing house, we demonstrate how variation margins affect the agents’ positions and indirect utilities, thereby generating a cost even if the counterparties face symmetrical collateral requirements. Moreover, we quantify the impact of the cost of collateral and suboptimal collateral rules on trading volume.\(^8\)

The paper is structured as follows. In the next section, we outline a simple continuous-time model with two agents and two sources of risk, one tradeable and one illiquid. We then provide some results for the private valuation of the illiquid position by agents with CARA preferences. In section 3, we introduce a risk sharing arrangement that mitigates the risk associated with the illiquid position, but gives rise to counterparty risk. We consider the case of no collateral, and fixed or contingent collateralization rules, offering insights into how the CSA can be optimally designed. Section 4 discusses some technical, yet fundamental, issues related to collateral rehypothecation, close-out conventions, and mark-to-market/model approaches to computing collateral amounts. Finally, section 5 concludes, while an appendix collect proofs and additional technical results.

2 Model

Consider two risk-averse agents, denoted by \(A\) and \(B\), with a trading horizon \([0,T]\), who have opposite exposure to a source of risk \(Z_T\) at the terminal date \(T\). Assume, for example, that party \(A\) is exposed to the payment of a random amount \(Z_T\) (a liability equal to \(-Z_T\)), whereas party \(B\) expects a random inflow equal to \(Z_T\). The dynamics

\(^8\)In a similar vein, Rampini \textit{et al.} (2011) demonstrate the impact of financing constraints on the use of collateralized risk management tools.
of \((Z_t)_{t \geq 0}\) is
\[ dZ_t = \sigma_Z dB^Z_t, \quad Z_0 = 0, \]
with \(B^Z\) a standard Brownian motion and \(\sigma_Z\) a positive volatility coefficient. Agents can trade in a financial market, where available are a money market account yielding the riskless rate \(r > 0\), and a risky security with price \(S\) evolving according to
\[ dS_t = \mu S_t dt + \sigma_S S_t dB^S_t, \quad S_0 = 1, \]
with \(\mu > r\) and \(\sigma_S > 0\). We assume that the Brownian motions \(B^S, B^Z\) are uncorrelated, meaning that tradable assets offer agents no way to span \(Z_T\), and the exposures to \(Z_T\) are illiquid until time \(T\), as they give rise to no intermediate dividends. The presence of an illiquid source of risk makes the market incomplete, and we need to take a stance on agents’ preferences to identify a valuation functional consistent with the absence of arbitrage opportunities. We assume agents to have both CARA utility
\[ u(x) = -\frac{1}{\gamma} e^{-\gamma x}, \]
with \(\gamma > 0\), and to optimize their utility from terminal wealth.

Agents are credit risky. We model their defaults exogenously, as triggered by the first jumps of two Poisson processes with the same parameter \(\lambda > 0\). We consider marginal transactions that have no direct impact on the firms’ default probability, hence the use of a tractable reduced form approach to credit risk. Abstracting from agents’ heterogeneity (safe for the opposite exposure to the illiquid source of risk) allows us to keep the paper focused and emphasize the role of counterparty risk in shaping collateral rules. Denoting by \(\tau^i\) the default time of party \(i \in \{A, B\}\), and by \(N_t = 1_{\tau^i \leq t}\) the default indicator process, we can express the dynamics of each agent’s wealth, \(W^i_t\), as
\[ dW^i_t = (1 - N^i_{t-}) \left[ \left( r W^i_t + \pi^i_t (\mu - r) \right) dt + \sigma^i_t dB^S_t \right] + N^i_t W^i_t r dt, \quad W^i_0 = w^i, \quad (2.1) \]
where $\pi_i^t$ is the wealth amount allocated by agent $i$ to the risky asset at each time $t \in [0, T \wedge \tau^i]$. As default is exogenous, we interpret the agents’ wealth as the resources dedicated to a specific transaction, or the balance of a trading account, which can turn negative and hence attract interest rate charges. For simplicity, we assume the latter to accrue at rate $r$. In the above, we have assumed that after default the residual wealth is invested in the money market account until maturity: $^9$ exclusion from the financial market is the agents’ default penalty (Alvarez and Jermann, 2000).

In the absence of credit risk mitigation tools, which will be introduced in the next section, agents solve the problem

$$\max_{\pi^i} E \left[ u(W^i_T + Z^i_T) \right], \quad (2.2)$$

subject to the budget constraint (2.1), where we have defined by $Z^i_T := (1_{i=B} - 1_{i=A})Z_T$ agent $i$’s exposure to $Z_T$, for $i \in \{A, B\}$. As $Z$ has symmetric distribution, agents achieve the same expected utility ex-ante. Their indirect utilities and optimal investment strategies can be computed explicitly, and are given in the next proposition.

**Proposition 2.1.** On $\{\tau^i > t\}$, each agent’s optimal value function is given by

$$v(t, W^i_t, Z^i_t) = -\frac{1}{\gamma} \exp \left( -\gamma e^{r(T-t)} W^i_t - \gamma Z^i_t + \gamma^2 \sigma^2 T - t \right) F(t), \quad i \in \{A, B\}, \quad (2.3)$$

$^9$An alternative approach is to assume that agents maximize their wealth at the terminal time $\tau \wedge T$ and simply ignore the residual liability. We make agents incur a default penalty to mimic the common practice of subjecting structuring desks to risk limits and counterparty risk charges.
with $\alpha = \frac{\lambda}{\lambda + 2 \frac{(\mu - r)^2}{\sigma^2_S}} \in (0, 1)$ and

$$F(t) = \left\{ (1 - \alpha) \exp \left( -\frac{1}{2} \left( \gamma^2 \sigma_Z^2 + \frac{(\mu - r)^2}{\sigma^2_S} \right) + \lambda \right) (T - t) \right\} + \alpha \exp \left( -\frac{1}{2} \gamma^2 \sigma_Z^2 (T - t) \right).$$

(2.4)

The optimal allocation to the risky asset is given by

$$\pi_t^* = \pi_i^* = \exp (-r(T - t)) \frac{\mu - r}{\gamma \sigma^2_S}. \tag{2.5}$$

The above shows that the optimal allocation to the risky asset is unaffected by the exposure to the illiquid payoff $Z_T$ and default risk, as they are both unspanned by the tradeable assets. The indirect utility $v$, however, takes into account both sources of risk. To understand how, it is useful to introduce the private valuation of the illiquid exposure as perceived by each agent.\textsuperscript{10} Using arguments similar to He and Pearson (1991), one can show that the private valuation of each agent, $V_t^i$, in our setting takes the explicit form

$$V_t^i = E_{i,t}^{i,*} [e^{-r(T - t)} Z_T^i] = e^{-r(T - t)} (Z_t^i - \gamma \sigma^2_S (T - t)). \tag{2.6}$$

Here, the conditional expectation $E_{i,t}^{i,*} [\cdot]$ represents the no-arbitrage pricing functional supported by agent $i$’s preferences, given the available trading opportunities and the budget constraint (2.1). Expression (2.3) can then be written as

$$v(t, W_t^i, Z_t^i) = -\frac{1}{\gamma} \exp \left( -\gamma e^{r(T - t)} (W_t^i + V_t^i) \right) F(t). \tag{2.7}$$

We then see that the indirect utilities take into account the certainty equivalent (at

\textsuperscript{10}See Detemple and Sundaresan (1999) for an extensive discussion of nontraded assets and their valuation.
maturity) of the agent’s aggregate resources and adjust it to reflect the risks and investment opportunities over the residual trading horizon. The factor $F(t)$ allows for the trading opportunities and the exposure to the illiquid source of risk and counterparty risk over $[t, T]$ ($(1 − α)$ addendum), and for the cost of being excluded from the market should default occur ($α$ addendum). The default penalty is endogenous, as it depends on the forgone investment opportunities and the residual exposure to the illiquid source of risk. As $λ$ goes to zero, $α$ vanishes, and the adjustment factor reduces to

$$F_{λ=0}(t) = \exp\left(-\frac{1}{2} \left( \gamma^2 \sigma_Z^2 + \frac{(μ - r)^2}{\sigma_S^2} \right) (T - t) \right),$$

(2.8)

thus recovering the results in Svensson and Werner (1993) and Teplà (2000).

The baseline case just outlined is quite restrictive. Because of risk aversion and the opposite exposure to $Z_T$, agents have natural risk sharing incentives. As a simple example, consider the case of a forward contract, i.e., an agreement whereby at time $T$ agent $A$ will transfer to agent $B$ a fixed fraction $k \geq 0$ of the realized exposure $Z_A^T$ upon payment of a price $p_k \geq 0$ agreed today. Ignoring counterparty risk for the moment, agent $A$ would have an incentive to enter the transaction as long as $U(W_A^T + (1 - k)Z_A^T - p_k) \geq U(W_A^T + Z_A^T)$, in which case we have $U(W_B^T + (1 - k)Z_B^T + p_k) \geq U(W_A^T + (1 - k)Z_A^T - p_k) \geq U(W_A^T + Z_A^T) = U(W_B^T + Z_B^T)$. From this we see that for $p_k = 0$ both agents would be indifferent between entering a long or short position in the forward contract, and would both obtain the same expected utility gains from solving problem (2.2) (with $Z_T$ now replaced by $(1 - k)Z_A^T$). We will therefore work with a zero forward price in the following.

While it is clear that full insurance ($k = 1$) would be optimal in the absence of counterparty risk, the same may not hold true when counterparty risk is explicitly
taken into account. To see this, note that each agent’s exposure to the illiquid source of risk is 
\((1 - k 1_{\tau > T})Z_T^i\) after entering a position in \(k\) units of the forward contract. As opposed to the previous case, the random variable 
\((1 - k 1_{\tau > T})Z_T^i\) and each agent’s terminal wealth are now correlated, either via the default penalty or via any collateral flows associated with a CSA. We will study the optimal demand for the forward contract in each separate case, and use the results to assess the impact of mandating suboptimal collateralization rules on the utilization of the hedging instrument.

3 Counterparty risk

As each counterparty can default on its obligations, let \(\tau = \min(\tau^A, \tau^B)\) denote the default time of the forward contract, and \(N_t := 1_{\tau \leq t}\) the associated indicator process. The agents’ budget constraints will take into account the risk of the counterparty not being able to honor the risk sharing agreement and, in the presence of a CSA, the cashflows associated with collateral flows (or other counterparty risk mitigation tools, such as put clauses or cancellability options). We denote by \(C^i\) the collateral process as seen from party \(i\), with the understanding that \(C^A = -C^B\). We assume collateral to be posted continuously and in cash.\(^{11}\) We let \(C_0\) be zero, and require \(C_t\) to be measurable with respect to the information available to agents an instant before each time \(t > 0\), in particular before a default may occur. Collateral amounts enter and exit the agents’ trading account, depending on the relative exposure of each agent to the counterparty and on the definition of collateral indicated by the CSA. Collateral amounts are fully fungible,\(^{11}\)

\(^{11}\)We ignore the choice of collateral type/quality in our setting. See Gorton and Metrick (2011) for an analysis of this important dimension in the repo market.
in the sense that they can be optimally invested by the agent in the financial market.\footnote{This means that there is an opportunity cost associated with posting collateral: this cost is endogenous in our model and related to the investment opportunities available in the market. The budget constraint (2.1) takes now the following form:\footnote{We denote $a^+ = \max\{0, a\}$ and $a^- = \max\{-a, 0\}$ for $a \in \mathbb{R}$.}\

\begin{equation}
\begin{aligned}
dW_i^t &= (1 - N_{i-}) \left[ (rW_i^t + \pi_i^t(\mu - r)) \, dt + \pi_i^t \sigma S_i \, dB_i^t \right] + N_{i-} W_i^t \, rdt \\
&+ (1 - N_{i-}) \left[ dC_i^t + \left( (R_{i-})^+ - C_{i-}^+ \right) \, dN_i^t - \left( (R_{i-})^- - C_{i-}^- \right) \, dN_i^t \right],
\end{aligned}
\end{equation}

with $W_0^i = w^i, i, j \in \{A, B\}, i \neq j$. In the above, $R^i$ denotes the replacement cost of the forward contract for agent $i$. The idea is that conditional on party $i$ ($j$) defaulting, agent $i$ would receive (pay) the full replacement cost of the forward contract from (to) the surviving counterparty, net of any collateral already held (posted). Note that in the case of overcollateralization, i.e., $C^+ > R^+$, provision of the full replacement cost may entail receiving any excess collateral posted. For ease of exposition, in this section we consider a common risk-free (and risk-neutral) close-out convention (e.g., Brigo et al., 2012b), meaning that we set\footnote{The term $R^i$ takes into account the exposure of party $i$ to the position in $k$ units of the forward contract, $-kZ_T^i$. In the case of $A$, for example, the agent faces a liability $-Z_T$ and therefore is long $k$ units of the forward contract, which will pay $kZ_T = -kZ_T^A$ at maturity in case of no default.}\

\begin{equation}
R_t^i = R_t^i(k) = E_t \left[ e^{-r(T-t)} \left( -kZ_T^i \right) \right] = -ke^{-r(T-t)} Z_t^i.
\end{equation}

Alternative close-out conventions, as well as the possibility to endogenize the replacement cost, are discussed in section 4.\footnote{The case of segregated collateral, discussed in section 4, would make collateral posting even more costly and thus strengthen our results.}
Agents will now consider the problem

$$\max_{(\pi^i,C^i)} E \left[ u \left( W^i_T + (1 - k1_{\tau>T}) Z^i_T - 1_{\tau>T} C^i_T \right) \right],$$

subject to the budget constraint (3.1), with the objective function taking into account the netting of the payoff from the forward contract and the terminal collateral amount.

We solve the problem at different levels of generality. We first study the case of no collateral, to understand the impact of counterparty risk (in addition to default risk and illiquidity) on the agents’ positions. We then consider two classes of admissible CSAs: those specifying fixed collateral rules, and those indicating contingent collateral rules.

### 3.1 No collateral

In the presence of counterparty risk, we need to consider three relevant cases at each time $t < T$ before the last default of the agents:

- Agent $i$ has already defaulted, $\{\tau^i \leq t < \tau^j\}$.
- The counterparty has already defaulted, $\{\tau \leq t < \tau^i\}$.
- No default has yet occurred, $\{\tau > t\}$.

The first case is uninteresting, as agent $i$’s wealth is simply invested in the money market account and any resources available at the terminal date $T$ will be used to meet the random exposure $Z_T$. In the second case, the agent is faced with the original exposure, without the help of a hedging instrument, and we are back to the case covered in Proposition 2.1. The third case is more interesting, as the indirect utility of each agent will reflect the trade-off between the risk sharing gains from the forward contract and the counterparty risk it gives rise to.
Proposition 3.1. Consider problem (3.3), with the collateral strategy restricted to the null process. On the event \( \{ \tau > t \} \), the optimal value function of agent \( i \in \{ A, B \} \) is given by

\[
v^d(t, W^i_t, Z^i_t) = -\frac{1}{\gamma} \exp \left( -\gamma e^{r(T-t)} (W^i_t + V^i_t) \right) F^d(t, Z^i_t; k), \tag{3.4}
\]

where the adjustment factor \( F^d \) solves PDE (A.5) reported in the appendix. The optimal allocation to the risky asset is again given by (2.5).

Comparing expression (3.4) with (2.3), we see that the agents’ indirect utilities have a similar structure, both considering the (uninsured) certainty equivalent of the agent’s aggregate resources at the terminal date, but applying different adjustment factors reflecting the hedging opportunities available. In (3.4), the adjustment factor depends on the agent’s exposure, as that is going to shape the recovery \( R^i \) in case default occurs in the next small time interval. In case of a null position in the forward contract, we clearly have \( F^d(t, Z^i_t; 0) = F(t) \).

3.2 Fixed collateralization

Consider the fully-fledged optimization problem (3.3), where collateral is forced to be expressed as a deterministic fraction of the risk-free close-out price (3.2). The space of admissible collateral rules is restricted to differentiable functions \( c^i : t \in [0, T] \rightarrow \mathbb{R}_+ \) satisfying \( c^i(0) = 0 \) and \( c^i = -c^j, i, j \in \{ A, B \} \). Hence the collateral amount held/posted by agent \( i \) at each time \( t < \tau \wedge T \) can be expressed as \( C^i_t = c(t)R^i_t \). It turns out that in this case problem (3.3) can be reduced to a deterministic control problem (see Eq. (A.6) with derivation in Appendix A.2). To obtain an intuition, it is useful to
consider some special cases that allow for explicit solutions.

**Proposition 3.2.** For \( r = 0 \), the optimal deterministic collateral rule, \((c_i^{t,*})\), satisfies:

\[
0 = \gamma \sigma Z \sqrt{t} (kc^{t,*}(t) - 1) \left[ h(t) + (1 - h(t)) \Phi \left( \gamma \sigma Z \sqrt{t} (kc^{t,*}(t) - 1) \right) \right] + (1 - h(t)) \varphi \left( \gamma \sigma Z \sqrt{t} (kc^{t,*}(t) - 1) \right),
\]

where

\[
h(t) = \alpha + (1 - \alpha) \exp \left\{ - \left( \frac{1}{2} \frac{\mu^2}{\sigma_S^2} + \lambda \right) (T - t) \right\},
\]

and \( \Phi(\cdot) \) and \( \varphi(\cdot) \) are the cdf and pdf of the standard Normal distribution, respectively.

In particular, \((3.5)\) implies that less than full collateralization is optimal for \( k = 1 \), i.e., \( c^t(t) < 1 \) for \( t < T \).

**Proposition 3.3.** The optimal collateral rule at the terminal time \( T \), \( c_T^* \), satisfies:

\[
\exp \left\{ \frac{1}{2} \gamma^2 \sigma_Z^2 T (k^2 - 2k) + \gamma^2 \sigma_Z^2 r k^2 X_1^*(T) \right\} r = \exp \left\{ \frac{1}{2} \gamma^2 \sigma_Z^2 k T c^*(T)(k c^*(T) - 2) + \gamma^2 \sigma_Z^2 r X_1^*(T) k^2 c^*(T) \right\}
\]

\[
\times \frac{(1 - c^*(T)) - k r \frac{X_1^*(T)}{T}}{k - 1 + r \frac{X_1^*(T)}{T}} \lambda,
\]

where \( X_1^*(T) = \int_0^T c^*(t) t \, dt \).

In particular, for \( k = 1 \) and \( r = 0 \), we have full collateralization in the sense that \( c^*(T) = 1 \) whereas for \( k = 1 \) and \( r > 0 \) we have less than full collateralization in the sense that \( c^*(T) < 1 \).

Hence, there exist two aspects that push the optimal collateral level away from full collateralization. On the one hand, if interest rates are positive, intermediate collateral
flows yield a risky terminal wealth level—and this risk generally cannot be mitigated as we assume that the collateral fraction is deterministic. On the other hand, even if the interest rate is zero, agents evaluate scenarios where collateral is paid differently than scenarios where collateral is received. More precisely, the party receiving the collateral still has the opportunity to invest in the financial market, whereas the defaulting party no longer has access to profitable investment opportunities. Thus marginal utilities differ in scenarios where collateral is paid relative to scenarios where collateral is received, rendering full collateralization suboptimal—even in the symmetric setting considered here.

A key question that arises in this context is the interaction between the risk sharing arrangement, that is the choice of $k$, and the collateral agreement $c^*$. We focus on the case $r = 0$, which is easier to analyze.

**Proposition 3.4.** Assume we impose a constant collateral rule $c(t) \equiv c \in [0, 1]$. Then we obtain the following for the optimal level of the risk sharing $k^*$:

1. For $c \in \{0, 1\}$, the optimal risk sharing level is $k^* = 1$.

2. For $c \in (0, 1)$, the optimal risk sharing level $k^* > 1$.

The intuition for the latter result is connected to the collateral level being a fraction of the recovery: The agent can improve her position by increasing $k$ as this implies she will receive a recovery in default states that is closer to the present value of the impending liability at the terminal time point, even if that is at the cost of an imperfect hedge in non-default states. For full collateralization, however, there is no need to push up the collateral level, whereas for no collateral at all increasing $k$ will only push the hedge to an imperfect level but does not affect collateral flows. This intuition carries through to
the optimal deterministic collateral schedule, which as we know from Proposition 3.2 features less than full collateral at intermediate times.

**Proposition 3.5.** For the optimal collateral schedule $c^*$, the optimal risk sharing arrangement $k^* > 1$.

Hence, sub-optimally imposing full collateralization, even though it results in what appears to be full risk sharing (in the sense that $k^* = 1$, which also emerges as the optimal solution in the absence of default risk), it actually impedes trading. The agents’ positions can be enhanced by jointly choosing optimal deterministic collateral and risk sharing arrangements that feature “less-than-full” collateral $c^* < 1$ and “more-than-full” risk sharing $k > 1$, respectively.

To illustrate these results, Figure 1 shows the optimal collateral schedule $c(\cdot)$ for base case parameters $(T, \gamma, \mu, r, \sigma_S, \sigma_Z, \lambda) = (1, 0.1, 0.02, 0, 0.2, 0.2, 0.05)$. The corresponding optimal level of risk sharing is $k^* = 1.0013$, which slightly exceeds one in line with Proposition 3.5.

![Figure 1: Optimal deterministic collateral fraction $c(\cdot)$. Parameter values: $(T, \gamma, \mu, r, \sigma_S, \sigma_Z, \lambda) = (1, 0.1, 0.02, 0, 0.2, 0.2, 0.05)$.](image-url)
As indicated in Proposition 3.2, we find that the optimal collateral fraction is less than one, i.e., we have less than full collateralization—though it is increasing over time and gets close to 1 for $t$ approaching $T = 1$. However, it does not quite approach 1, but rather $c(1) = 0.9987 = 1/k^*$. The intuition is just as above: The agent chooses $k > 1$ to trade off optimal cash flows between default and non-default states, but then the optimal collateral faction at $t = T$ is also reduced exactly to produce an “optimal” recovery of $k^* c(T) = 1$ if default occurs right before the contract expires.

As a consequence, trading activity increases in the default probability since a higher default probability implies a larger weight of default states. To illustrate, Figure 2 displays the optimal risk sharing level $k^*$ as a function of $\lambda$ between 0.05 and 0.55 and the other parameters as in the base case. We observe that while the level increases, even for relatively high default probabilities the “excessive” risk sharing (beyond what is optimal in the absence of default risk) is still below 1%.

Finally, Figure 3 shows some sensitivities of the optimal collateral schedule and the optimal level of risk sharing when varying key parameters. The bottom panels...
display sensitivities with respect to capital market parameters. As the risk premium increases, i.e. as either $\mu$ increases or $\sigma_S$ decreases, the opportunity cost of posting collateral increases—in the sense that it shifts resources from my default states to my counterparty’s default states—and thus the optimal collateral fraction decreases relative to the base case. As a consequence, the optimal risk sharing level increases from 1.0013 to 1.0027 and 1.0021, respectively. Indeed this effect may even lead to negative collateral levels early in the contract term (we cut off negative levels at zero to facilitate the comparison between the collateral schedules).

![Figure 3: Optimal deterministic collateral fraction $c(\cdot)$. Parameter values in the base case: $(T, \gamma, \mu, r, \sigma_S, \sigma_Z, \lambda) = (1, 0.1, 0.02, 0, 0.2, 0.2, 0.05)$.](image)

Similarly, when decreasing the volatility of the OTC instrument $\sigma_Z$, we observe a reduction of the optimal collateral level. Here, we have a reduction of the risk due to the liability while the risk of defaulting and the associated default penalty remain the same.
Hence, the aforementioned opportunity cost of posting collateral also drives collateral down and, consequently, the optimal risk sharing level up. In contrast, for a higher risk aversion level $\gamma$, the motive for smoothing resources between default states and non-default states is increased, resulting in a higher collateral fraction and a reduced trading level $k^* = 1.0003$ relative to the based case.

### 3.3 Contingent collateralization

We now generalize the previous example, by allowing for collateral rules that may be contingent on the path of the state variable vectors $(W^i, Z^i, C^i)$. We take the class of the admissable collateral rules to include processes with continuous paths that are predictable with respect to the information generated by the state vectors. For each time $t \in [0, T \land \tau)$, we define the collateral amount posted by party $i$ as

$$C^i_t = \int_0^t c^i(s, W^i_s, Z^i_s, C^i_s) dZ^i_s,$$

and further require $c^i$ to satisfy the following symmetry conditions:

$$
\begin{align*}
C^i_t &= -C^j_t, \\
c^i(t, W^i_t, Z_t, C^i_t) &= -c^j(t, W^j_t, Z^j_t, C^j_t) = -c^j(t, -W^i_t, -Z^i_t, -C^i_t).
\end{align*}
$$

The optimal solution now entails:

**Proposition 3.6.** Consider problem (3.3), with $C^i$ optimally chosen in the space of admissable contingent collateral rules defined above. Then, the optimal collateralization strategy is independent of wealth, $c^i^*(t, W^i_t, Z^i_t, C^i_t) = \bar{c}^i(t, Z^i_t, C^i_t)$, and results in a positive collateral fraction on $(0, T \land \tau)$. 

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So similarly to before, collateral is positive as it mitigates counterparty risk. To obtain some more insights on the form of the optimal collateral rule, we derive numerical results by solving the Hamilton-Jacobi-Bellman (HJB) equation associated with the optimal stochastic control problem.\textsuperscript{15}

Figure 4 shows the optimal collateralization fraction $\hat{c}^i$ for different times to maturity and for the average value of $(Z^i, C^i)$, whereas Figures 5 depicts the surface representing the optimal collateral rule $\hat{c}^i$ for different configurations of the pair $(Z^i, C^i)$. The findings show similarities to the deterministic collateral case treated in the previous section. Specifically, optimal collateral levels are again positive, but lower than ‘full’ collateralization would require. This comes as no surprise since a similar intuition prevails: The counterparties evaluate scenarios where collateral is paid and where collateral is received differently, and the difference in marginal utilities pushes the levels away from full collateralization. Interestingly, the collateral levels do vary between different states $(Z^i, C^i)$. A possible explanation is that unlike the deterministic collateral case, state contingent collateral flows allow the agents to mitigate the exposure to fluctuations in the terminal wealth induced by collateral flows.

\textsuperscript{15}Because of the symmetry requirements (3.6), the HJB for this problem gives rise to a PDE with distributed arguments that needs a suitable extension of standard finite difference methods for boundary value problems.
Figure 4: Optimal collateral fraction for average value of the pair \((Z, C^i)\) and for different times to maturity. Parameter values: \(\gamma = 0.2, T = 1, \lambda = 5\%, k = 0.9, \sigma_Z = 20\%, r = 3\%, \mu = 6\%, \sigma_S = 20\%\).

Figure 5: Optimal collateral fraction \(\hat{c}^i\) as a function of the collateral posted to date, \(C^i\), and realization of \(Z\). Parameter values as in figure 4.
4 Discussion

4.1 Re-hypothecation of collateral

In the presentation of the model in sections 2 and 3, we have always assumed that collateral amounts received from a counterparty become part of the trading account and can therefore be invested in the financial market. More generally, collateral could be re-pledged for other purposes, potentially exposing the counterparty to delays in collateral transfers, should the position turn in her favour, and to close-out risk, in case excess collateral was posted and used to support trades difficult to unwind. For these reasons collateral may be segregated and its re-hypothecation forbidden, meaning that the term \((1 - N_t) dC_t\) would not appear in the budget dynamics (3.1), as collateral would be posted in a separate account managed by a custodian. Our setting can easily accommodate these changes and provide similar results. However, \textit{ceteris paribus} the optimal collateral fractions would be uniformly lower than in the case of re-hypothecation, as the benefits from holding collateral would now be limited to mitigating counterparty risk (rather than generating excess returns from investment of additional resources in the financial market), while the costs of posting collateral would be unaffected by segregation in this setting.

4.2 Full collateralization

It is clear from the results of the previous section that full collateralization is in general a suboptimal strategy. The fact that such strong requirement is common in some OTC markets (e.g., in interest rate swaps; see ISDA, 2010b), should be interpreted in the light of frictions affecting the collateralization process in real-world transactions. In practice
the collateralization process is discrete and, depending on the specific asset class or products considered, collateral revisions may be relatively infrequent. In this situation full collateralization may provide a simple way to provide a buffer against gaps in the protection against counterparty risk originating from discrete collateralization.

There is an additional important issue to note, however. The term ‘full collateralization’ is often a misnomer in OTC transactions, as counterparty risk mitigation is ‘full’ only to the extent of the replacement cost defined by the CSA.\textsuperscript{16} The definition is not unique, and often ambiguous, as there may be cases when liquidators are given the opportunity to choose between a risk-free close-out, such as the one considered in (3.2), and a credit risky close-out, for example estimated from a range of quotes obtained in the market after the default event. Even if the definition were unambiguous, it is clear that collateralization would be full only if the replacement cost were to coincide with the market-value of the OTC contract being considered. This is often not the case, as CSAs would typically specify proxies and models to be used to determine collateral amounts, which may be only broadly correlated with the value of the contract under consideration.

5 Conclusion

In this work we have considered OTC transactions subject to bilateral default risk. In our baseline setting agents are exposed to a nontradable source of risk that can be hedged by entering a risk sharing agreement with a counterparty that is credit risky. Counterparty

\textsuperscript{16}A different situation is when the maximum payout from an instrument can be defined, and an equal amount of collateral is posted at inception and segregated (this is the case for catastrophe bonds and other asset backed securities; e.g., Lakdawalla and Zanjani, 2012). Even there, however, collateralization is not entirely full, as cash collateral is too expensive and high-yield (but credit-risky) fixed-income instruments are typically used.
risk can be mitigated by suitably designing a CSA indicating when and how much collateral to post during the life of the transaction. We have determined optimal collateral rules over different admissible strategies, therefore providing a microeconomic foundation for collateralization strategies observed in practice. At the same time, we have developed a framework that allows one to address several issues related to the delicate features of OTC transactions, such as re-hypothecation and segregation of collateral, the role of funding costs, and the definition of close-out conventions. Our setting shows that funding costs, which in our setting arise endogenously as the opportunity cost of optimally allocating resources to a trading technology, are an integral part of the optimal CSA design. Moreover, we have developed a framework that can be used to quantify the effects of suboptimal collateral rules, and collateral scarcity and segregation, on the use of OTC derivatives.

References


Brigo, D., A. Pallavicini and D. Perini (2012c). Funding, collateral and hedging: uncovering the mechanics and the subtleties of funding valuation adjustments. Available at SSRN 2161528.


Paper.


A Technical Appendix

A.1 Results Without Collateral

Proof of Proposition 2.1. Consider the position of agent $A$:

$$v(t, W_t, Z_t) = \mathbb{E}_t \left[ \int_t^T e^{-\lambda(s-t)} \lambda U(W_s e^{r(T-s)} - Z_T) ds \right] + \mathbb{E}_t \left[ e^{-\lambda(T-t)} U(W^A_T - Z_T) \right]$$

$$= -\mathbb{E}_t \left[ \int_t^T e^{-\lambda(s-t)} \frac{\lambda}{\gamma} e^{-\gamma W_s e^{r(T-s)} e^{\gamma Z_T}} ds \right] - \frac{1}{\gamma} \mathbb{E}_t \left[ e^{-\lambda(T-t)} e^{-\gamma W^A_T} e^{\gamma Z_T} \right]$$

$$= -\frac{\lambda}{\gamma} e^{\frac{1}{2} \sigma_2^2 (T-t) + \gamma Z_t} \mathbb{E}_t \left[ \int_t^T e^{-\lambda(s-t)} e^{-\gamma W_s e^{r(T-s)}} ds \right] - \frac{1}{\gamma} \mathbb{E}_t \left[ e^{-\lambda(T-t)} e^{-\gamma W^A_T} \right],$$

where $v(T, x, z) = -\frac{1}{\gamma} e^{-\gamma(x-z)}$. Define

$$\tilde{v}(t, W_t) := e^{-\frac{1}{2} \gamma^2 \sigma_2^2 (T-t) - \gamma Z_t - \lambda t} v(t, W_t, Z_t)$$

$$= -\frac{\lambda}{\gamma} e^{\frac{1}{2} \gamma^2 \sigma_2^2 (T-t) - \gamma Z_t} \mathbb{E}_t \left[ \int_t^T e^{-\lambda s - \gamma W_s e^{r(T-s)}} ds \right] - \frac{1}{\gamma} \mathbb{E}_t \left[ e^{-\lambda T - \gamma W^A_T} \right],$$

where $\tilde{v}(T, x) = -\frac{1}{\gamma} e^{-\lambda T - \gamma x}$. Hence, the HJB equation for this problem is

$$\tilde{v}_t + \sup_{\pi} \left\{ -\frac{\lambda}{\gamma} e^{-\lambda t - \gamma x e^{r(T-t)}} + r x \tilde{v}_x + x \pi (\mu - r) \tilde{v}_x + \frac{1}{2} x^2 \sigma_2^2 \tilde{v}_{xx} \right\} = 0.$$  

From the first order condition, we obtain

$$\pi^* = -\frac{(\mu - r)}{\sigma_2^2} \frac{\tilde{v}_x}{x \tilde{v}_{xx}}.$$
Therefore, the HJB equation becomes

\[
\begin{aligned}
\tilde{v}_t - \frac{\lambda}{\gamma} e^{-\lambda t - \gamma x e^{r(T-t)}} + r x \tilde{v}_x - \frac{1}{2} \frac{(\mu-r)^2}{\sigma^2} \tilde{v}_{xx} &= 0, \\
\tilde{v}(T, x) &= -\frac{1}{\gamma} e^{-\lambda T - \gamma x}.
\end{aligned}
\] (A.1)

The solution for (A.1) is:

\[
\tilde{v}(t, x) = -\frac{1}{\gamma} e^{-\gamma x e^{r(T-t)}} \left( e^{\frac{1}{2} \frac{(\mu-r)^2}{\sigma^2}(T-t)} (1 - \alpha) e^{-\lambda T} + \alpha e^{-\lambda t} \right),
\] (A.2)

where

\[
\alpha = \frac{\lambda}{\lambda + \frac{1}{2} \frac{(\mu-r)^2}{\sigma^2}}.
\]

And therefore,

\[
\pi^*_t X_t = \frac{\mu - r}{\gamma \sigma^2} e^{-r(T-t)}.
\] (A.3)

In addition,

\[
v(t, x, z) = e^{\frac{1}{2} \gamma^2 \sigma^2 Z(T-t) + \gamma z + \lambda t} \left( -\frac{1}{\gamma} e^{-\gamma x e^{r(T-t)}} \left( (1 - \alpha) e^{\frac{1}{2} \frac{(\mu-r)^2}{\sigma^2}(T-t) - \lambda T} + \alpha e^{-\lambda t} \right) \right).
\]

Thus,

\[
v(t, x, z) = -\frac{1}{\gamma} \exp \left\{ -\gamma x e^{r(T-t)} + \gamma z + \frac{1}{2} \gamma^2 \sigma^2 Z(T-t) \right\} \times \left( (1 - \alpha) e^{\frac{1}{2} \frac{(\mu-r)^2}{\sigma^2}(T-t) - \lambda (T-t)} + \alpha \right),
\] (A.4)

where

\[
\alpha = \frac{\lambda}{\lambda + \frac{1}{2} \frac{(\mu-r)^2}{\sigma^2}}.
\]
Proof of 3.1. Again, consider the position of agent $A$. For $t < \tau := \min\{\tau^A, \tau^B\}$:

\[
v(t, W_t, Z_t) = \mathbb{E}_t \left[ U^A \left( W_T - Z_T + kZ_T 1_{\{\tau > T\}} \right) \right]
\]

\[
= \mathbb{E}_t \left[ \int_t^T e^{-2\lambda(s-t)} \lambda \mathbb{E}_s \left[ U^A \left( (W_s + kZ_s^+)e^{r(T-s)} - Z_T \right) \right] ds \right] 
+ \mathbb{E}_t \left[ \int_t^T e^{-2\lambda(s-t)} \lambda \lambda v^{(1)}(s, W_s - kZ_s, Z_s) ds \right] 
+ \mathbb{E}_t \left[ e^{-2\lambda(T-t)} U^A \left( W^A_t + (k-1)Z_T \right) \right]
\]

where $v^{(1)}$ is defined as in Proposition 2.1. Thus,

\[
v(t, W_t, Z_t) = -\frac{\lambda}{\gamma} \mathbb{E}_t \left[ \int_t^T e^{-2\lambda(s-t)} e^{-\gamma(W_s+kZ_s^+)e^{r(T-s)}+\gamma Z_s + \frac{\gamma^2}{2} \sigma^2_s (T-s)} ds \right] 
-\frac{\lambda}{\gamma} \mathbb{E}_t \left[ \int_t^T e^{-2\lambda(s-t)} e^{-\gamma(W_s-kZ_s^-)e^{r(T-s)}+\gamma Z_s + \frac{\gamma^2}{2} \sigma^2_s (T-s)} \times \right.
\]

\[
\times \left( (1-\alpha) e^{-\frac{1}{2}(\mu-r)^2(T-s)-\lambda(T-s)} + \alpha \right) ds \right]
-\frac{1}{\gamma} \mathbb{E}_t \left[ e^{-2\lambda(T-t)} e^{-\gamma W_T - \gamma (k-1)Z_T} \right]
\]

\[\Rightarrow e^{-2\lambda t} v(t, W_t, Z_t) = -\frac{\lambda}{\gamma} \mathbb{E}_t \left[ \int_t^T e^{-2\lambda s} e^{-\gamma Z_s + \frac{\gamma^2}{2} \sigma^2_s (T-s)} \times \right.
\]

\[
\times \left( e^{-\gamma(W_s+kZ_s^+)e^{r(T-s)}} + \alpha e^{-\gamma(W_s-kZ_s^-)e^{r(T-s)}} \right.
\]

\[
+ (1-\alpha) e^{-\gamma(W_s-kZ_s^-)e^{r(T-s)}-\frac{1}{2}(\mu-r)^2(T-s)-\lambda(T-s)} \left) ds \right]
-\frac{1}{\gamma} \mathbb{E}_t \left[ e^{-2\lambda T} e^{-\gamma W_T - \gamma (k-1)Z_T} \right]
\]
Hence, the function \( \tilde{v}(t, x, z) := e^{-2\lambda t} v(t, x, z) \) satisfies the HJB equation

\[
\tilde{v}_t - \frac{\lambda}{\gamma} e^{-2\lambda t + \gamma z} e^{(T-t)} \left( e^{-\gamma (x+k z)} e^{r(T-t)} + (1 - \alpha) e^{-\gamma (x-k z)} e^{r(T-t)} - \frac{1}{2} \sigma_S^2 (T-t) - \frac{1}{2} (\mu - r)^2 \right) \\
+ \alpha e^{-\gamma (x-k z)} e^{r(T-t)} \\
+ \sup_{\pi} \left\{ r x \tilde{v}_x + \pi (x-r) \tilde{v}_x + \frac{1}{2} x^2 \pi^2 \sigma_S^2 \tilde{v}_{xx} + \frac{1}{2} \sigma_Z^2 \tilde{v}_{zz} \right\} = 0,
\]

where

\[
\tilde{v}(T, x, z) = -\frac{1}{\gamma} e^{-\gamma x - \gamma (1-k) z - 2\lambda T}.
\]

From the first order condition, we obtain

\[
\pi^* = -\frac{(\mu - r)}{\sigma_S^2} \frac{\tilde{v}_x}{x \tilde{v}_{xx}}.
\]

Hence, the HJB equation becomes

\[
\tilde{v}_t - \frac{\lambda}{\gamma} e^{-2\lambda t + \gamma z} e^{(T-t)} \left( e^{-\gamma (x+k z)} e^{r(T-t)} + (1 - \alpha) e^{-\gamma (x-k z)} e^{r(T-t)} - \frac{1}{2} \sigma_S^2 (T-t) - \frac{1}{2} (\mu - r)^2 \right) \\
+ \alpha e^{-\gamma (x-k z)} e^{r(T-t)} \\
+ r x \tilde{v}_x - \frac{1}{2} \frac{(\mu - r)^2}{\sigma_S^2} \frac{\tilde{v}_x^2}{\tilde{v}_{xx}} + \frac{1}{2} \sigma_Z^2 \tilde{v}_{zz} = 0,
\]

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or equivalently,

$$\tilde{v}_t - \frac{\lambda}{\gamma} \exp \left\{ -2\lambda t + \gamma z + \frac{1}{2} \gamma^2 \sigma_Z^2 (T - t) - \gamma (x + k z^+) e^{r(T-t)} \right\}$$

$$- \frac{(1 - \alpha)\lambda}{\gamma} \exp \left\{ -2\lambda t + \gamma z + \frac{1}{2} \gamma^2 \sigma_Z^2 (T - t) - \gamma (x - k z^-) e^{r(T-t)} - \left( \frac{(\mu - r)^2}{\sigma_S^2} + \lambda \right) (T - t) \right\}$$

$$- \frac{\alpha \lambda}{\gamma} \exp \left\{ -2\lambda t + \gamma z + \frac{1}{2} \gamma^2 \sigma_Z^2 (T - t) - \gamma (x - k z^-) e^{r(T-t)} \right\}$$

$$+ r x \tilde{v}_x - \frac{1}{2} \left( \frac{\mu - r}{\sigma_S^2} \right) \tilde{v}_x^2 + \frac{1}{2} \sigma_Z^2 \tilde{v}_{xx} = 0,$$

Then, $\tilde{v}$ has the form (for generic $i \in \{A, B\}$ now)

$$\tilde{v}(t, x, z^i) = -\frac{1}{\gamma} \exp \left\{ -2\lambda t - \gamma \left( e^{r(T-t)} x + z^i - \gamma \sigma_Z^2 (T - t) \right) \right\} F^c(t, z^i),$$

where

$$0 = F^c_t - \left( 2\lambda + \gamma^2 \sigma_Z^2 + \frac{1}{2} \left( \frac{\mu - r}{\sigma_S^2} \right)^2 \right) F^c - \sigma_Z^2 \gamma F^c_{z^i} - \frac{1}{2} \sigma_Z^2 F^c_{z^i z^i}$$

$$+ \lambda \exp \left( \gamma e^{r(T-t)} k(z^i)^+ \right)$$

$$+ \lambda \exp \left( -\gamma e^{r(T-t)} k(z^i)^- \right) \left[ \alpha + (1 - \alpha) \exp \left( -\frac{1}{2} \left( \frac{\mu - r}{\sigma_S^2} + \lambda \right) (T - t) \right) \right],$$

with boundary condition $F^c(T, z^i; k) = \frac{1}{\lambda} \exp(-\gamma k z^i)$.

\[\square\]

A.2 Optimal Fixed Collateralization

Assume $C_t = c_t k e^{-r(T-t)} Z_t$, where $c.$ is a differentiable function. It follows—just like in the more general case treated in Appendix A.3—that the optimal investment strategy is given by

$$\pi_t^* = \frac{(\mu - r)}{\sigma_S^2 \gamma} e^{-r(T-t)},$$

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and we have for agent’s A wealth on \( t < \tau := \min\{\tau^A, \tau^B\} \):

\[
dW^i_t = r W^i_t \, dt + \pi^*_t \left[ (\mu - r) \, dt + \sigma_S \, dB^S_t \right] + \left( r c_t \, e^{-r(T-t)} Z^i_t \, dt + c'_t \, e^{-r(T-t)} Z^i_t \, dZ^i_t \right).
\]

**Lemma A.1.**

\[
W^i_t = \exp \{ rt \} \left[ W_0 + \frac{(\mu - r)^2}{\sigma^2} \gamma e^{-rT} \, t + \frac{\mu - r}{\sigma \gamma} e^{-rT} B^S_t - \int_0^t e^{-rT} \, k c_s Z^i_s \, ds - e^{-rT} \, k c_t Z^i_t \right].
\]

**Proof.** By Itô’s formula, we have:

\[
dW^i_t = r W^i_t \, dt + \pi^*_t \left[ (\mu - r) \, dt + \sigma \, dB^S_t \right] - c'_t \, e^{-r(T-t)} Z^i_t \, dt - c_t \, e^{-r(T-t)} \, dZ^i_t.
\]

Without loss of generality, we focus on agent A. Then:

\[
V_0 = E \left[ u \left( W^A_T + (k 1_{T > T} - 1) Z_T \right) \right] = -E \left[ \int_0^T e^{-2 \lambda s} \frac{\lambda}{\gamma} E_s \left[ \exp \left\{ -\gamma \left( W^A_s + k e^{-r(T-s)} (Z^+_s - c_s Z^+_s) \right) e^{r(T-s)} + \gamma Z_T \right\} \right] \right] \, ds
\]

\[
- \int_0^T e^{-2 \lambda s} \frac{\lambda}{\gamma} \exp \left\{ -\gamma \left( W^A_s - k e^{-r(T-s)} (Z^-_s - c_s Z^-_s) \right) e^{r(T-s)} + \gamma Z_s + \frac{1}{2} \gamma^2 \sigma^2 (T - s) \right\} \times \left\{ \alpha + (1 - \alpha) \exp \left\{ -\left( \frac{1}{2} \left( \frac{(\mu - r)^2}{\sigma^2} + \lambda \right) (T - s) \right) \right\} \right\} \, ds
\]

\[
- \frac{1}{\gamma} E \left[ e^{-2 \lambda T} \exp \left\{ -\gamma W_T - \gamma (k - 1) Z_T + \gamma k c_T Z_T \right\} \right],
\]

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where we used Proposition 2.1. Therefore:

\[
\begin{align*}
E \left[ u \left( W_T^3 + (k 1_{r > T} - 1) Z_T \right) \right] &= - \int_0^T e^{-2 \lambda s} \frac{\lambda}{\gamma} E \left[ \exp \left\{ - \gamma \left( W_0 e^{rT} + \frac{(\mu - r)^2}{\sigma_S^2} s + \frac{(\mu - r)}{\sigma_S} B^S_T + \int_0^s r \, k_c u Z_u \, du \right) \right. \\
& \quad \left. \left. - \gamma k Z_s^+ - \gamma k c_s (Z_s - Z_s^-) + \gamma Z_s + \frac{1}{2} \gamma^2 \sigma_Z^2 (T - s) \right) \right\} \\
& \quad \exp \left\{ - \gamma \left( W_0 e^{rT} + \frac{(\mu - r)^2}{\sigma_S^2} s + \frac{(\mu - r)}{\sigma_S} B^Z_T + \int_0^s r \, k_c u Z_u \, du \right) \right. \\
& \quad \left. \left. + \gamma k Z_s^- - \gamma k c_s (Z_s + Z_s^-) e^{r(T-s)} + \gamma Z_s + \frac{1}{2} \gamma^2 \sigma_Z^2 (T - s) \right) \right\} \\
& \quad \times \left\{ \alpha + (1 - \alpha) \exp \left\{ - \left( \frac{1}{2} \left( \frac{(\mu - r)^2}{\sigma_S^2} + \lambda \right) (T - s) \right) \right\} \right\} \, ds \\
& - \frac{1}{\gamma} e^{-2 \lambda T} E \left[ \exp \left\{ - \gamma W_0 e^{rT} - \frac{(\mu - r)^2}{\sigma_S^2} T - \frac{(\mu - r)}{\sigma_S} B^S_T \right. \\
& \quad \left. - \gamma \int_0^T r \, e^{r(T-u)} k_c u Z_u \, du - \gamma k c_T Z_t - (k-1) Z_T + \gamma k c_T Z_T \right\} \right] \\
& = - \frac{1}{\gamma} e^{-2 \lambda T} \left[ \int_0^T e^{-2 \lambda s} \frac{\lambda}{\gamma} E \left[ \exp \left\{ - \frac{(\mu - r)^2}{\sigma_S^2} s + \frac{(\mu - r)}{\sigma_S} B^S_T \right. \\
& \quad \left. - \gamma k Z_s^+ + \gamma Z_s + \gamma k c_s Z_s^- \right\} \right. \\
& \quad \left. \times \left( \alpha + (1 - \alpha) \exp \left\{ - \left( \frac{1}{2} \left( \frac{(\mu - r)^2}{\sigma_S^2} + \lambda \right) (T - s) \right) \right\} \right) \right\} \, ds \\
& \quad + e^{-2 \lambda T} E \left[ \exp \left\{ - \frac{(\mu - r)^2}{\sigma_S^2} T - \frac{(\mu - r)}{\sigma_S} B^S_T - \gamma (k-1) Z_T - \gamma r \int_0^T k_c u Z_u \, du \right\} \right] \\
& = - \frac{1}{\gamma} e^{-2 \lambda T} \left[ \int_0^T e^{-2 \lambda s} \frac{\lambda}{\gamma} \exp \left\{ - \frac{1}{2} \frac{\lambda}{\gamma} \left( \frac{(\mu - r)^2}{\sigma_S^2} s + \frac{1}{2} \lambda \gamma^2 \sigma_Z^2 (T - s) \right) \right. \\
& \quad \left. \left( \frac{\lambda}{\gamma} \exp \left\{ - \gamma r \int_0^s k_c u Z_u \, du - \gamma k Z_s^+ + \gamma k c_s Z_s^- Z_s + \gamma Z_s \right\} \right) \right. \\
& \quad \left. \left. \times h(s) \right) \right\} \, ds \\
& + e^{-2 \lambda T} \left[ \exp \left\{ - \frac{1}{2} \frac{\lambda}{\gamma} T \right. \frac{\lambda}{\gamma} \exp \left\{ - \gamma (k-1) Z_T - \gamma r \int_0^T k_c u Z_u \, du \right\} \right. \\
& \quad \left. \right) \right\} \right]\right]
\end{align*}
\]

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where

\[ h(s) = \alpha + (1 - \alpha) \exp \left( -\left( \frac{1}{2} \frac{(\mu - r)^2}{\sigma^2_S} + \lambda \right) (T - s) \right). \]

We need the following two lemmas:

**Lemma A.2.** For \( Z \sim N(0, 1) \) standard Normal, we have:

\[
\mathbb{E} [\exp\{a Z + b Z\}] = e^{\frac{1}{2}a^2} \Phi(a) + e^{\frac{1}{2}b^2} (1 - \Phi(b)),
\]

where \( \Phi(\cdot) \) is the standard Normal cumulative distribution function.

**Lemma A.3.**

\[
\int_0^s \left( \int_v^s c_u \, du \right)^2 \, dv = 2 \int_0^s \int_0^u c_u c_v \, dv \, du.
\]

The proofs follow by simple calculations.

Now define:

\[
N_1^s = \int_0^s c_u \, dB_u^Z \quad \text{and} \quad N_2^s = B_u^S.
\]

Then \( (N_1^s, N_2^s)' \) is normal with expected value zero and covariance matrix:

\[
\begin{pmatrix}
\int_0^s (\int_v^s c_u \, du)^2 \, dv & \int_0^s c_u u \, du \\
\int_0^s c_u u \, du & s
\end{pmatrix}.
\]
Hence, with \( \text{ith} Z_1 \) and \( Z_2 \) standard normal, we can write:

\[
\begin{align*}
E \left[ \exp \left\{ -\gamma r \int_0^s k c_u Z_u \, du - \gamma k Z_s^+ + \gamma k c_s Z_s^- + \gamma Z_2 \right\} \right] \\
= E \left[ \exp \left\{ -\gamma k r \sigma Z \sqrt{\int_0^s \left( \int_v^s c_u \, du \right)^2 \, dv} \times \left[ \frac{\int_0^s c_u \, du}{\sqrt{\int_0^s \left( \int_v^s c_u \, du \right)^2 \, dv}} Z_1 \right. \right. \right. \\
\left. \left. \left. \left. + \left( 1 - \frac{\left( \int_0^s c_u \, du \right)^2}{\int_0^s \left( \int_v^s c_u \, du \right)^2 \, dv} \right)^{\gamma/2} Z_2 \right] k \gamma \sigma Z \sqrt{s} Z_1^+ + k \gamma \sigma_Z c_s \sqrt{s} Z_1^- + \gamma \sigma_Z \sqrt{s} Z_1 \right\} \right] \\
= \exp \left\{ \frac{1}{2} \gamma^2 k^2 r^2 \sigma_Z^2 \left( \int_0^s \left( \int_v^s c_u \, du \right)^2 \, dv - \frac{1}{s} \left( \int_0^s c_u \, du \right)^2 \right) \right\} \\
\times E \left[ \exp \left\{ \left( -\gamma k r \sigma_Z \frac{\int_0^s c_u \, du \, dv}{\sqrt{s}} + \gamma \sigma_Z \sqrt{s} - \gamma k \sqrt{s} \right) Z_1^+ \right. \right. \\
\left. \left. - \left( -\gamma k r \sigma_Z \frac{\int_0^s c_u \, du \, dv}{\sqrt{s}} + \gamma \sigma_Z \sqrt{s} - \gamma k \sqrt{s} Z_1 \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right.
Analogously, we obtain:

\[
E\left[ \exp\left\{-\gamma r \int_0^s k c_u Z_u \, du - \gamma c_s Z_s^+ + \gamma k Z_s^- + \gamma Z_s \right\} \right]
\]

\[
= \exp\left\{ \frac{1}{2} \gamma^2 k^2 r^2 \sigma_Z^2 \left( \int_0^s c_u \int_0^u c_v \, dv \, du - \int_0^s c_u \int_0^u c_v \, dv \, du \right) \right\}
\]

\[
\times \left[ \exp\left\{ \frac{1}{2} \left( -\gamma k r^2 \sigma_Z \int_0^s \frac{c_u \, du}{\sqrt{s}} + \gamma \sigma_Z \sqrt{s} - \gamma k \sigma_Z c_s \sqrt{s} \right)^2 \right\}
\]

\[
\times \Phi \left( -\gamma k r \sigma_Z \int_0^s \frac{c_u \, du}{\sqrt{s}} + \gamma \sigma_Z \sqrt{s} - \gamma k \sigma_Z c_s \sqrt{s} \right)
\]

\[
+ \exp\left\{ \frac{1}{2} \left( -\gamma k r \sigma_Z \int_0^s \frac{c_u \, du}{\sqrt{s}} + \gamma \sigma_Z \sqrt{s} - \gamma k \sigma_Z \sqrt{s} \right)^2 \right\}
\]

\[
\times \left( 1 - \Phi \left( -\gamma k r \sigma_Z \int_0^s \frac{c_u \, du}{\sqrt{s}} + \gamma \sigma_Z \sqrt{s} - \gamma k \sigma_Z \sqrt{s} \right) \right).
\]

And similarly:

\[
E \left[ \exp\left\{-\gamma (k - 1) Z_T - \gamma r \int_0^T k c_u Z_u \, du \right\} \right]
\]

\[
= E \left[ \exp\left\{-\gamma r \sigma_Z k \sqrt{\int_v^T \left( \int_v^u c_u \, du \right)^2 \, dv \left[ \frac{\int_v^T c_u \, du}{\sqrt{\int_v^T \left( \int_v^u c_u \, du \right)^2 \, dv \, T}} \right]} Z_1 \right\}
\]

\[
+ \left( 1 - \frac{\left( \int_v^T c_u \, du \right)^2}{\int_v^T \left( \int_v^u c_u \, du \right)^2 \, dv \, T} \right) \right)^{\frac{1}{2}} Z_2 \right\} - \gamma \sigma_Z (k - 1) \sqrt{T} Z_1 \right\}
\]

\[
\left[ \exp\left\{ \frac{1}{2} \gamma^2 k^2 r^2 \sigma_Z^2 \left( \int_0^T c_u \int_0^u c_v \, dv \, du - \int_0^T c_u \int_0^u c_v \, dv \, du \right) \right\}
\]

\[
\times \exp\left\{ \frac{1}{2} \left( (k - 1) \sqrt{T} + k r \frac{\int_v^T c_u \, du}{\sqrt{T}} \right)^2 \right\} \right].
\]
Thus, putting these three parts into $V_0$, we obtain:

$$V_0 = -\frac{1}{\gamma} \exp \left\{ -\gamma W_0 e^{T} \right\} \left[ \int_0^T e^{-2\lambda s} \lambda \exp \left\{ \frac{1}{2} \frac{(\mu - r)^2}{\sigma_S^2} s \right. \right. \right.$$  

$$+ \frac{1}{2} \gamma^2 \sigma_Z^2 (T - s) + \gamma^2 r^2 \kappa^2 \sigma_Z^2 \int_0^s \frac{c_u s - u}{s} \int_0^u c_v v d v d u \} \right.$$  

$$\left. \exp \left\{ \frac{1}{2} \left( \gamma \sigma_Z \sqrt{s} - \gamma k \gamma \sigma_Z \int_0^s \frac{c_u u d u}{\sqrt{s}} - k \gamma \sigma_Z \sqrt{s} \right)^2 \right\} \right.$$  

$$\left[ \phi \left( \gamma \sigma_Z \sqrt{s} - k \gamma \sigma_Z \int_0^s \frac{c_u u d u}{\sqrt{s}} - k \gamma \sigma_Z \sqrt{s} \right) \left[ 1 - h(s) \right] + h(s) \right]$$  

$$+ \exp \left\{ \frac{1}{2} \left( \sigma_Z \sqrt{s} - \gamma \sigma_Z \int_0^s \frac{c_u u d u}{\sqrt{s}} - k \gamma \sigma_Z \sqrt{s} \right)^2 \right\}$$  

$$\left[ h(s) + [1 - h(s)] - [1 - h(s)] \phi \left( \gamma \sigma_Z \sqrt{s} - k \gamma \sigma_Z \int_0^s \frac{c_u u d u}{\sqrt{s}} - k \gamma \sigma_Z \sqrt{s} \right) \right] \right\} ds$$  

$$+ e^{-2\lambda T} \exp \left\{ \frac{1}{2} \frac{(\mu - r)^2}{\sigma_S^2} T + \gamma^2 r^2 \kappa^2 \sigma_Z^2 \int_0^T \frac{c_u T - u}{T} \int_0^u c_v v d v d u \} \right.$$  

$$\exp \left\{ \frac{1}{2} \gamma^2 \sigma_Z^2 \left( (k - 1) \sqrt{T} + r \kappa \int_0^T \frac{c_u u d u}{\sqrt{T}} \right)^2 \right\} \right\}$$  

$$= -\frac{1}{\gamma} \exp \left\{ -\gamma W_0 e^{T} \right\} \left[ \int_0^T e^{-2\lambda s} \lambda \exp \left\{ \frac{1}{2} \frac{(\mu - r)^2}{\sigma_S^2} s + \frac{1}{2} \gamma^2 \sigma_Z^2 (T - s) + \frac{1}{2} \gamma^2 \sigma_Z^2 s \right. \right. \right.$$  

$$+ \frac{k^2 \gamma^2 r^2 \sigma_Z^2}{\sigma_S^2} \int_0^s \frac{c_u u}{s} \int_0^u c_v v d v d u - \gamma^2 \sigma_Z^2 k r \int_0^s c_u u d u \} \right.$$  

$$\times \left( \exp \left\{ k^2 \gamma^2 \sigma_Z^2 r \int_0^s c_u u d u - \gamma^2 \sigma_Z^2 k s + \frac{1}{2} \gamma^2 \kappa^2 \sigma_Z^2 s \right. \right. \right.$$  

$$\left. \left. \right\} \left[ h(s) + [1 - h(s)] \phi \left( \gamma \sigma_Z \sqrt{s} - k \gamma r \sigma_Z \int_0^s \frac{c_u u d u}{\sqrt{s}} - \gamma \sigma_Z \sqrt{s} \right) \right] \right. \right.$$  

$$+ \exp \left\{ k^2 \gamma^2 \sigma_Z^2 r \int_0^s c_u u d u c_s - \gamma^2 \sigma_Z^2 k s c_s + \frac{1}{2} \gamma^2 \kappa^2 \sigma_Z^2 c_s \right. \right.$$  

$$\left. \left. \right\} \left[ h(s) + [1 - h(s)] \phi \left( -\gamma \sigma_Z \sqrt{s} + \gamma k r \sigma_Z \int_0^s \frac{c_u u d u}{\sqrt{s}} + \gamma \sigma_Z \sqrt{s} \right) \right] \right\} ds$$  

$$+ e^{-2\lambda T} \exp \left\{ \frac{1}{2} \frac{(\mu - r)^2}{\sigma_S^2} T + \frac{1}{2} \gamma^2 \sigma_Z^2 (k - 1)^2 T + \gamma^2 r^2 \kappa^2 \sigma_Z^2 \int_0^T \frac{c_u T - u}{T} \int_0^u c_v v d v d u \} \right.$$  

$$\exp \left\{ \gamma^2 \sigma_Z^2 k^2 r^2 \int_0^T \frac{c_u u}{T} \int_0^u c_v v d v d u + \gamma^2 \sigma_Z^2 (k - 1) k r \int_0^T c_u u d u \right\} \right\}$$  

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Reordering, we obtain:

\[
V_0 = -\frac{1}{\gamma} \exp \left( -\gamma W_0 e^{rT} + \frac{1}{2} \gamma^2 \Sigma Z T \right) \\
\int_0^T e^{-2\lambda s} \lambda \exp \left\{ -\frac{1}{2} \frac{(\mu - r)^2}{\sigma^2} s + k^2 \gamma^2 r^2 \int_0^s c_u \int_0^u c_v \, dv \, du \\
- \gamma^2 \Sigma Z k r \int_0^s e^{-r(T-u)} c_u \, du \right\} \\
\times \left[ \exp \left\{ \gamma^2 \Sigma Z \left( \frac{1}{2} k^2 c_s k s - k s c_s + k^2 r \int_0^s c_u \, du \right) \right\} \\
\times \left[ h(s) + [1 - h(s)] \Phi \left( k \gamma \Sigma Z \sqrt{s} - \gamma r \Sigma Z \frac{\int_0^s c_u \, du}{\sqrt{s}} - \gamma \Sigma Z \sqrt{s} \right) \right] \\
\exp \left\{ \gamma^2 \Sigma Z \left( \frac{1}{2} k^2 c_s k s - k s c_s + k^2 r \int_0^s c_u \, du \right) \right\} \\
\times \left[ h(s) + [1 - h(s)] \Phi \left( k \gamma \Sigma Z c_s \sqrt{s} - \gamma r \Sigma Z \frac{\int_0^s c_u \, du}{\sqrt{s}} - \gamma \Sigma Z \sqrt{s} \right) \right] \right] \\
+ e^{-2\lambda T} \exp \left\{ -\frac{1}{2} \frac{(\mu - r)^2}{\sigma^2} T + \frac{1}{2} \gamma^2 \Sigma Z T (1 - 2k) \right\} \\
\times \exp \left\{ \gamma^2 \Sigma Z \left( r^2 k^2 \int_0^T c_u \int_0^u c_v \, dv \, du + (k - 1) k r \int_0^T c_u \, du \right) \right\} ,
\]

and the optimization problem is maximizing this problem in \( c \).

To formulate this problem as a dynamic program, let \( X(t) = (x_1(t), x_2(t))' \) with

\[
x_1(t) = \int_0^t c_u \, du \\
x_2(t) = \int_0^t c_u \int_0^u c_w \, dw \, du
\]

so that

\[
X'(t) = \begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} c_t t \\ c_t \int_0^t c_w \, dw \end{pmatrix} = \begin{pmatrix} c_t t \\ c_t x_1(t) \end{pmatrix}.
\]

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Then, ignoring constant factors, we can represent the optimization problem as:

\[
\begin{align*}
\min_{c} \int_0^T g(X(t), c_t) \, dt + h(X(T)) \\
\text{s.th. } X'(t) &= \begin{pmatrix} t c_t \\ c_t x_1(t) \end{pmatrix}
\end{align*}
\] (A.6)

where

\[
g(X(t), c_t) = e^{-2\lambda t} \lambda \exp\left\{ -\frac{1}{2} \left( \frac{\mu - r}{\sigma^2} - 2 \sigma^2 t + k^2 \frac{\gamma^2 r^2}{\sigma^2} x_2(t) - \gamma^2 \frac{2 r}{\sigma^2} k x_1(t) \right) \right\} \times \left[ h(t) + [1 - h(t)] \Phi \left( \gamma \frac{\sigma Z t}{\sqrt{t}} - \frac{\gamma \sigma Z}{\sqrt{t}} \right) \right]
\]

and

\[
h(X(T)) = e^{-2\lambda T} \exp\left\{ -\frac{1}{2} \left( \frac{\mu - r}{\sigma^2} T + 2 \frac{\gamma^2 \sigma^2 T}{\sigma^2} (1 - 2k) + \gamma^2 \frac{2 r}{\sigma^2} x_2(T) + (k^2 - k) r x_1(T) \right) \right\}.
\]

The Hamilton-Jacobi-Bellman Equation for the optimal value function reads:

\[
0 = \frac{\partial}{\partial t} V(t, X(t)) + \min_{c_t} \{ g(X(t), c_t) + \nabla_X V(t, X(t)) X'(t) \} ; V(T, X(T)) = h(X(T)). \quad (A.7)
\]

**Proof of Proposition 3.2.** In case \( r = 0 \), we have a state independent problem, and a necessary
condition for the optimal collateral rule at time $t$ amounts to optimizing the function $g$ where:

\[
g(c_t) = e^{-2\lambda t} \lambda \exp \left\{-\frac{1}{2} \frac{\mu^2}{\sigma^2 S} t \right\} \times \left( \exp \left\{ \frac{1}{2} \gamma^2 \sigma^2_Z t (k^2 - 2k) \right\} \times \left[ h(t) + [1 - h(t)] \Phi \left( \frac{\gamma \sigma_Z \sqrt{t}}{1 - k} \right) \right] + \exp \left\{ \frac{1}{2} \gamma^2 \sigma^2_Z t (k^2 c_t^2 - k t c_t) \right\} \left[ h(t) + [1 - h(t)] \Phi \left( \frac{\gamma \sigma_Z \sqrt{t} (k c_t - 1)}{1 - h(t)} \right) \right] \right).\]

Taking first order condition yields:

\[
0 = \frac{1}{2} \gamma^2 \sigma^2_Z t (k^2 c_t - 2k) \exp \left\{ \frac{1}{2} \gamma^2 \sigma^2_Z t (k^2 c_t^2 - 2k c_t) \right\} \times \left[ h(t) + [1 - h(t)] \Phi \left( \frac{\gamma \sigma_Z \sqrt{t} (k c_t - 1)}{1 - h(t)} \right) \right] + \exp \left\{ \frac{1}{2} \gamma^2 \sigma^2_Z t (k^2 c_t^2 - 2k c_t) \right\} \gamma \sigma_Z \sqrt{t} k \varphi \left( \frac{\gamma \sigma_Z \sqrt{t} (k c_t - 1)}{1 - h(t)} \right) [1 - h(t)],
\]

which simplifies to (3.5) completing the proof. \qed

**Proof of Proposition 3.3.** At the terminal time, the first order condition in the HJB Equation (A.7) reads:

\[
0 = \lambda e^{-2\lambda T} \exp \left\{ -\frac{1}{2} \frac{(\mu - r)^2}{\sigma^2 S} T + k^2 \gamma^2 r^2 \sigma^2_Z x_2(T) - \gamma^2 \sigma^2_Z k r x_1(T) \right\} \times \gamma^2 \sigma^2_Z (c_T k^2 T - k T + k^2 r \sigma^2_Z x_2(T)) \exp \left\{ \gamma^2 \sigma^2_Z \left( \frac{1}{2} k^2 c_T^2 T - k T c_T + k^2 r x_1(T) c_T \right) \right\} + e^{-2\lambda T} (k^2 - k) \gamma^2 \sigma^2_Z \exp \left\{ -\frac{1}{2} \frac{(\mu - r)^2}{\sigma^2 S} T + \frac{1}{2} \gamma^2 \sigma^2_Z T (k^2 - 2k) + \gamma^2 \sigma^2_Z (k^2 r^2 x_2(t) + (k^2 - k) \\
+ e^{-2\lambda T} k^2 \gamma^2 \sigma^2_Z \exp \left\{ -\frac{1}{2} \frac{(\mu - r)^2}{\sigma^2 S} T + \frac{1}{2} \gamma^2 \sigma^2_Z T (k^2 - 2k) + \gamma^2 \sigma^2_Z (k^2 r^2 x_2(t) + (k^2 - k) r x_1(t) \right\} \exp \left\{ \frac{1}{2} \gamma^2 \sigma^2_Z T (k^2 - 2k) + \gamma^2 \sigma^2_Z r x_1(T) k^2 \right\} \exp \left\{ \frac{1}{2} \gamma^2 \sigma^2_Z T (k^2 - 2k) + \gamma^2 \sigma^2_Z r x_1(T) k^2 \right\} \right. \left( k - 1 + k r \frac{x_1(T)}{T} \right) \end{align*}

\[
\Leftrightarrow 0 = \lambda \left( c_T k - 1 + k r \frac{x_1(T)}{T} \right) \exp \left\{ \frac{1}{2} \gamma^2 \sigma^2_Z T (k c_T - 2) + \gamma^2 \sigma^2_Z r x_1(T) k^2 \right\} \exp \left\{ \frac{1}{2} \gamma^2 \sigma^2_Z T (k^2 - 2k) + \gamma^2 \sigma^2_Z r x_1(T) k^2 \right\} \right. \left( k - 1 + k r \frac{x_1(T)}{T} \right)
\]

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so that

\[
\exp \left\{ \frac{1}{2} \gamma^2 \sigma^2_Z T (k^2 - 2k) + \gamma^2 \sigma^2_Z r x_1(T) k^2 \right\} r
= \exp \left\{ \frac{1}{2} \gamma^2 \sigma^2_Z k T c_T (k c_T - 2) + \gamma^2 \sigma^2_Z r x_1(T) k^2 c_T \right\} \times \frac{1 - c_T k - k r x_1(T)}{k - 1 + r k x_1(T)}
\]

which completes the proof.

**Proof of Proposition 3.4 and 3.5.** We have:

\[
\frac{\partial V_0}{\partial k} = \int_0^T e^{-2 \lambda s} \lambda \exp \left\{ -\frac{1}{2} \frac{\mu^2}{\sigma^2_S} \right\} \left[ \gamma^2 \sigma^2_Z s (k - 1) \exp \left\{ \frac{1}{2} \gamma^2 \sigma^2_Z s (k^2 - 2k) \right\} \left[ h(s) + [1 - h(s)] \Phi \left( \gamma \sigma_Z \sqrt{s} (1 - k) \right) \right] 
+ \gamma^2 \sigma^2_Z s (k c_s^2 - c_s) \exp \left\{ \frac{1}{2} \gamma^2 \sigma^2_Z s (k^2 c_s^2 - 2k c_s) \right\} \left[ h(s) + [1 - h(s)] \Phi \left( \gamma \sigma_Z \sqrt{s} (k c_s - 1) \right) \right] \right] ds 
+ e^{-2 \lambda T} \exp \left\{ -\frac{1}{2} \frac{\mu^2}{\sigma^2_S} T \right\} \gamma^2 \sigma^2_Z T (k - 1) \exp \left\{ \frac{1}{2} \gamma^2 \sigma^2_Z T (k^2 - 2k) \right\}
\]

since the other derivatives cancel, which shows Proposition 3.4.

Now assume optimal collateralization, then

\[
c_s \gamma^2 \sigma^2_Z s (k c_s - 1) \exp \left\{ \frac{1}{2} \gamma^2 \sigma^2_Z s (k^2 c_s^2 - 2k c_s) \right\} \left[ h(s) + [1 - h(s)] \Phi \left( \gamma \sigma_Z \sqrt{s} (k c_s - 1) \right) \right] 
= -c_s \gamma \sigma_Z \sqrt{s} \left( \gamma \sigma_Z \sqrt{s} (k c_s - 1) \right) [1 - h(s)] \exp \left\{ \frac{1}{2} \gamma^2 \sigma^2_Z s (k^2 c_s^2 - 2k c_s) \right\} 
= -c_s \gamma \sigma_Z \left[ 1 - h(s) \right] \exp \left\{ -\frac{1}{2} \frac{\mu^2}{\sigma^2_S} s \right\}
\]

which completes the proof.
so that:

\[
(k - 1) \int_0^T e^{-2\lambda s} \lambda \exp \left\{ -\frac{1}{2} \frac{\mu^2}{\sigma_S^2} s \right\} \exp \left\{ \frac{1}{2} \frac{\gamma^2 \sigma_Z^2}{s} (k^2 - 2k) \right\} [h(s) + [1 - h(s)] \Phi \left( \gamma \sigma_Z \sqrt{s} (k c_s - e^{-2\lambda T} \exp \left\{ -\frac{1}{2} \frac{\mu^2}{\sigma_S^2} T \right\} T \exp \left\{ \frac{1}{2} \frac{\gamma^2 \sigma_Z^2}{s} T (k^2 - 2k) \right\}\right] \gamma \sigma_Z
\]

\[
= \int_0^T c_s [1 - h(s)] \exp \left\{ -\frac{1}{2} \frac{\gamma^2 \sigma_Z^2}{s} s \right\} ds
\]

which shows that \( k > 1 \). \( \square \)

### A.3 Optimal Dynamic Collateralization

With collateral we have:

\[
v(t, W_t, Z_t) = \mathbb{E}_t \left[ U^A \left( W_T - k Z_T + Z_T 1_{\{\tau > T\}} \right) \right]
\]

\[
= \mathbb{E}_t \left[ \int_t^T e^{-2\lambda(s-t)} \lambda \mathbb{E}_s \left[ U^A((W_s + Z_s^+ - C_s^+)e^{(T-s)} - kZ_T) \right] ds \right]
\]

\[
+ \mathbb{E}_t \left[ \int_t^T e^{-2\lambda(s-t)} \lambda \mathbb{V}^{(1)}(s, W_s - Z_s^- + C_s^-)ds \right]
\]

\[
+ \mathbb{E}_t \left[ e^{-2\lambda(T-t)} U^A(W_T^A + (1 - k)Z_T) \right]
\]

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where $v^{(1)}$ is defined as in Proposition 3.1. Thus,

$$e^{-2\lambda t} v(t, W_t, Z_t) = -\frac{\lambda}{\gamma} E_t \left[ e^{-2\lambda s} \int_0^T e^{-\gamma (W_s + Z_s^+ - C_s^+)} e^{r(T-s) + \gamma k Z_s t + \frac{1}{2} \gamma^2 k^2 \sigma_Z^2 (T-s) ds} \right]$$

$$-\frac{\lambda}{\gamma} E_t \left[ e^{-2\lambda s} \int_0^T e^{-\gamma (W_s - Z_s^- + C_s^-)} e^{r(T-s) + \gamma k Z_s t + \frac{1}{2} \gamma^2 k^2 \sigma_Z^2 (T-s) \times} \right]$$

$$\times \left( (1 - \alpha) e^{-\frac{1}{2} \frac{(\mu - r)^2}{\sigma^2_S} (T-s) - \lambda (T-s) + \alpha} ds \right)$$

$$-\frac{1}{\gamma} E_t \left[ e^{-2\lambda T} e^{-\gamma (1-k) Z_T} \right]$$

$$= -\frac{\lambda}{\gamma} E_t \left[ e^{-2\lambda s} \int_0^T e^{-\gamma (W_s + Z_s^+ - C_s^+) e^{r(T-s) + \alpha e^{-\gamma (W_s - Z_s^- + C_s^-) e^{r(T-s)}}}} \right]$$

$$\times \left( (1 - \alpha) e^{-\gamma (W_s - Z_s^- + C_s^-) e^{r(T-s)} - \frac{1}{2} \frac{(\mu - r)^2}{\sigma^2_S} (T-s) - \lambda (T-s)) \right) ds$$

$$-\frac{1}{\gamma} E_t \left[ e^{-2\lambda T} e^{-\gamma (1-k) Z_T} \right]$$

Hence, the function $\tilde{v}(t, x, z) := -\gamma e^{-2\lambda t} v(t, x, z)$ satisfies the HJB equation

$$0 = \tilde{v}_t + \sup_{\pi} \left\{ x \pi (\mu - r) \tilde{v}_x + \frac{1}{2} x^2 \pi^2 \sigma_Z^2 \tilde{v}_{xx} \right\}$$

$$+ \sup_{c(t)} \left\{ \lambda \exp \left\{ -2\lambda t + \gamma k z + \frac{1}{2} \gamma^2 k^2 \sigma_Z^2 (T-t) - \gamma (x + [1 - c(t)] z^+) e^{r(T-t)} \right\} \right.$$}

$$+ (1 - \alpha) \lambda \exp \left\{ -2\lambda t + \gamma k z - \gamma (x - [1 - c(t)] z^-) e^{r(T-t)} \right.$$}

$$+ \left( \frac{1}{2} \gamma^2 k^2 \sigma_Z^2 - \frac{1}{2} \frac{(\mu - r)^2}{\sigma^2_S} - \lambda \right) (T-t) \right\}$$

$$+ \alpha \lambda \exp \left\{ -2\lambda t + \gamma k z + \frac{1}{2} \gamma^2 k^2 \sigma_Z^2 (T-t) - \gamma (x - [1 - c(t)] z^-) e^{r(T-t)} \right\}$$

$$\times + r x \tilde{v}_x + c'(t) z \tilde{v}_x + \frac{1}{2} (c(t))^2 \sigma_Z^2 \tilde{v}_{xx} + \frac{1}{2} \sigma_Z^2 \tilde{v}_{xz} + c(t) \sigma_Z^2 \tilde{v}_{xz} \right\},$$

where

$$\tilde{v}(T, x, z) = e^{-\gamma x - \gamma (1-k) z - 2\lambda T}.$$
From the first order condition, we obtain

$$\pi^* = -\frac{(\mu - r)}{\sigma_S^2} \frac{\tilde{v}_x}{x \tilde{v}_{xx}}.$$

Thus, we have that

$$\tilde{v}(t, x, z, y) = -\gamma e^{-2\lambda t} v(t, x, z, y)$$

satisfies the HJB equation:

$$0 = \tilde{v}_t + r x \tilde{v}_x - \frac{1}{2} \frac{(\mu - r)^2}{\sigma_S^2} \tilde{v}_{xx} + \frac{1}{2} \sigma_Z^2 \tilde{v}_{zz}$$

$$+ \lambda \exp \left\{-2\lambda t + \gamma k z + \frac{1}{2} \gamma^2 k^2 \sigma_Z^2 (T - t) - \gamma (x - y^+) e^{r(T - t)} - \gamma z^+ \right\}$$

$$+ (1 - \alpha) \lambda \exp \left\{-2\lambda t + \gamma k z - \gamma (x + y^-) e^{r(T - t)} + \gamma z^- \right\}$$

$$+ \alpha \lambda \exp \left\{-2\lambda t + \gamma k z + \frac{1}{2} \gamma^2 k^2 \sigma_Z^2 (T - t) - \gamma (x + y^-) e^{r(T - t)} + \gamma z^- \right\}$$

$$+ \inf \left\{ \frac{1}{2} c \sigma_Z^2 \tilde{v}_{xx} + c \sigma_Z^2 \tilde{v}_{xz} + \frac{1}{2} c \sigma_Z^2 \tilde{v}_{yy} + c \sigma_Z^2 \tilde{v}_{yz} \right\}$$

where

$$\tilde{v}(T, x, z, y) = e^{-\gamma x - \gamma (1-k)z - 2\lambda T + \gamma y}.$$ 

Moreover, we have that the optimal \(c^*\) is given by

$$c^* = \frac{-\tilde{v}_{xz}(t, x, z, y) - \tilde{v}_{xx}(t, x, -z, -y) - \tilde{v}_{yz}(t, x, z, y) - \tilde{v}_{y}(t, x, -z, -y)}{\tilde{v}_{xx}(t, x, z, y) + \tilde{v}_{xx}(t, x, -z, -y) + \tilde{v}_{yy}(t, x, z, y) + \tilde{v}_{yy}(t, x, -z, -y) + 2 \tilde{v}_{xy}(t, x, z, y) + 2 \tilde{v}_{xy}(t, x, -z, -y)}$$

(A.8)

Then, \(\tilde{v}\) has the form

$$\tilde{v}(t, x, z) = -\frac{\lambda}{\gamma} \exp \left\{-2\lambda t - \gamma x e^{r(T - t)} + \frac{1}{2} \gamma^2 k^2 \sigma_Z^2 (T - t) \right\} \Theta(t, z, y),$$

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where

\[
\Theta_t = \left( 2\lambda + \frac{1}{2} \gamma^2 k^2 \sigma_Z^2 + \frac{1}{2} \frac{(\mu - r)^2}{\sigma_S^2} \right) \Theta + \frac{1}{2} \sigma_Z^2 \Theta_{zz} \quad (A.9)
\]

\[
-\gamma \exp \left\{ \gamma k z + \gamma y^+ e^{r(T-t)} - \gamma z^+ \right\} \\
-(1-\alpha)\gamma \exp \left\{ \gamma k z - \gamma y^- e^{r(T-t)} + \gamma z^- - \frac{1}{2} \frac{(\mu - r)^2}{\sigma_S^2} (T-t) - \lambda(T-t) \right\} \\
-\alpha \gamma \exp \left\{ \gamma k z - \gamma y^- e^{r(T-t)} + \gamma z^- \right\} \\
+ \frac{1}{2} (c^*)^2 \sigma_Z^2 \left( \gamma^2 e^{2r(T-t)} \Theta - 2\gamma e^{r(T-t)} \Theta_y + \Theta_{yy} \right) - c^* \sigma_Z^2 \left( \gamma e^{r(T-t)} \Theta_z - \Theta_{yz} \right) = 0,
\]

and

\[
\Theta(T, z, y) = -\frac{\gamma}{\lambda} e^{-\gamma(1-k)z+y}.
\]

Furthermore,

\[
c^* = \frac{e^{r(T-t)} \Theta_z(t, z, y) + e^{r(T-t)} \Theta_z(t, -z, -y) - \frac{1}{2} \Theta_{yz}(t, z, y) - \frac{1}{2} \Theta_{gz}(t, -z, -y) + \frac{1}{\gamma} (\Theta_{yy}(t, z, y) + \Theta_{yy}(t, -z, -y)) - 2e^{r(T-t)} (\Theta_y(t, z, y) + \Theta_y(t, -z, -y))}{\gamma e^{2r(T-t)} (\Theta(t, z, y) + \Theta(t, -z, -y)) + \frac{1}{\gamma} (\Theta_{yy}(t, z, y) + \Theta_{yy}(t, -z, -y)) - 2e^{r(T-t)} (\Theta_y(t, z, y) + \Theta_y(t, -z, -y))} \quad (A.10)
\]