A “Coefficient of Variation” for Skewed and Heavy-Tailed Insurance Losses

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Abstract

We propose a new scalar risk measure that broadens the “spread-to-shift” ratio of the coefficient of variation (CV) to insurance loss distributions characterized by substantial skewness and/or heavy tails. The extended coefficient of variation (ECV), a cosine-based measure, is derived by maximizing the marginal Shannon information associated with the Fourier transform of a loss distribution’s probability density function. The ECV is particularly appropriate for portfolios of skewed and/or heavy-tailed insurance losses because it possesses a closed-form expression for members of the Lévy-stable family of distributions, to which all portfolios of i.i.d. individual losses must converge.

Keywords: Coefficient of variation, risk measure, skewness, heavy tails, Fourier transform, Shannon information, Lévy-stable family.

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1. Introduction

Numerous scalar (i.e., one-dimensional) measures of “risk” have been proposed for the evaluation of insurance losses. The simplest and most commonly used measures are the standard deviation (SD) and variance, which contain equivalent information because the latter is just the square of the former. A more sophisticated measure is the coefficient of variation (CV), which normalizes the SD by dividing it by the corresponding mean. This “spread-to-shift”\(^2\) ratio is particularly useful in comparing the risks associated with individual losses with widely different means or portfolios of losses with widely different numbers of exposures.

A major shortcoming of the CV is that it does not account for substantial skewness and/or heavy tails associated with the relevant loss distribution. In fact, the CV’s numerator fails to be defined for heavy-tailed losses;\(^3\) for example, if \(X \sim \text{Pareto-Lomax}(\lambda, \xi)\) (i.e.,
\[
f_X(x) = \frac{\lambda \xi^k}{(x + \xi)^{k+1}}, \quad x \in (0, \infty),\]
\(^4\) then the SD (and therefore the CV) does not exist for \(\lambda \leq 2\). As a consequence, actuaries often turn to any of a number of skewness- and tail-sensitive risk measures, including: (i) value at risk (VaR), (ii) tail value at risk, (iii) excess tail value at risk, (iv) expected policyholder deficit, and (v) default value. As shown by Powers (2007), each of these measures can be viewed as a special case of the expression
\[
E_X \left[ aX + bu \mathbb{1}_{X > cv[X]} \right] \cdot \Pr \{ X > dw[X] \},
\]

\(^2\) Throughout the article, we use the terms “spread” and “shift” somewhat informally. The former term indicates a measure of a loss’ unnormalized variation based upon a penalty function that is symmetric about some specific point, and the latter term indicates a measure of a loss’ likely position in its sample space.
\(^3\) By “heavy-tailed,” we mean that a loss does not possess a finite variance.
\(^4\) Pareto-Lomax distributions are also sometimes called “Pareto (II)” in the research literature.
where \(a, b, c,\) and \(d\) are constants and \(u[X],\) \(v[X],\) and \(w[X]\) are either means or percentiles of the distribution of \(X.\)\(^5\)

Unfortunately, expression (1) is not well defined if \(X\) is sufficiently heavy-tailed that it does not possess a finite mean (for example, if \(X \sim \text{Pareto-Lomax}(\lambda, \xi)\) with \(\lambda \leq 1\)). Consequently, the only commonly used risk measure that works in the case of a badly behaved mean is the purely percentile-based VaR, given by \(F_x^{-1}(1 - \varepsilon)\) (for any selected tail probability \(\varepsilon\)). The insensitivity of VaR to pathologies of the mean undoubtedly explains some of this risk measure’s popularity in the insurance and finance literatures. However, knowing VaR for only one (or even a few) fixed tail probabilities \(\varepsilon\) leaves much to be desired in terms of characterizing the overall risk associated with the random variable \(X\). Percentiles can tell us much about one tail of the distribution, but little or nothing about the center or other tail.

In the present work, we propose a new scalar risk measure – the extended coefficient of variation (ECV) – that broadens the concept of a CV to insurance losses characterized by substantial skewness and/or heavy tails. The new risk measure is derived from a cosine-based analysis that involves maximizing the marginal Shannon information associated with the Fourier transform of the distribution’s probability density function (PDF). This approach is particularly appropriate for portfolios of skewed and/or heavy-tailed insurance losses because the ECV possesses a closed-form expression for members of the Lévy-stable family of distributions,\(^6\) to

\[^5\] Specifically, for risk measure (i), we set \(a = c = d = 0, b = 1,\) and \(u[X] = F_x^{-1}(1 - \varepsilon)\) for some tail probability \(\varepsilon;\) for (ii), \(a = c = 1, b = d = 0,\) and \(v[X] = F_x^{-1}(1 - \varepsilon);\) for (iii), \(a = c = 1, b = d = 0,\) and \(v[X] = E_x[X];\) for (iv), \(a = 1, b = -(\text{Net Worth})/E_x[X], c = (\text{Net Worth})/E_x[X], d = 0,\) and \(u[X] = v[X] = E_x[X];\) and for (v), \(a = 1, b = -(\text{Net Worth} + \text{Net Income})/E_x[X], c = d = (\text{Net Worth} + \text{Net Income})/E_x[X],\) and \(u[X] = v[X] = w[X] = E_x[X].\)

\[^6\] Lévy-stable distributions are also sometimes called “stable,” “stable Paretian,” or “Pareto-Lévy” in the research literature. They have been proposed as the most appropriate model for asset returns by Mandelbrot (1963) and others.
which all portfolios of i.i.d. individual losses must converge. The ECV simplifies to a constant multiple of the CV for members of the Gaussian (normal) family.

2. Spread and Shift

In developing a novel risk measure, one must be very clear about what it should capture. For a continuous loss random variable \( X \) with finite variance, the CV is written formally as

\[
CV = \frac{\left(E[X |X - \mu|^2]\right)^{\frac{1}{2}}}{\mu},
\]

where the numerator represents the spread (i.e., a symmetric measure of the loss' variation – the SD – in the same units as the original random variable) and the denominator represents the shift (i.e., a measure of the loss' likely position – the mean – also in the same units). One interesting observation about this ratio is that the shift in the denominator serves as the central point of symmetry for the spread in the numerator. More rigorously, one can say that, among all possible choices \( m \) for the point of symmetry in the numerator, the denominator minimizes the value of the resulting spread; that is,

\[
\mu = \arg\inf_{m \in \mathbb{R}} \left\{ E[X |X - m|^2]^{\frac{1}{2}} \right\}.
\]

Taken together, equations (2) and (3) immediately suggest the following generalized coefficient of variation (GCV):

\[
GCV_g = \frac{g^{-1}\left(E_X g\left(|X - m_g|\right)\right)}{m_g}.
\]

where

\[
m_g = \arg\inf_{m \in \mathbb{R}} \left\{ g^{-1}\left(E_X g\left(|X - m|\right)\right) \right\}.
\]
for any function \( g(\cdot) \) that is increasing in a neighborhood to the right of 0.

One simple special case of (4), which itself includes the CV as a special case, is based upon the power function, \( g(y) = y^p \), for \( p \in [0, \infty) \). This family offers the potential of addressing heavy-tailed random variables because \( E_x \left[ |X - m_p|^p \right] \) exists as long as the tails of the relevant PDF diminish faster than \( 1/|x|^{p+1} \) (e.g., if \( X \sim \text{Pareto-Lomax}(\lambda > p, \xi) \)). However, no single power-based measure,

\[
CV_p = \left( \frac{E_x \left[ |X - m_p|^p \right]}{m_p} \right)^{\frac{1}{p}},
\]

where

\[
m_p = \arg \inf_{m \in \mathbb{R}} \left\{ \left( E_x \left[ |X - m|^p \right] \right)^{\frac{1}{p}} \right\},
\]

can be applied usefully to all random variables \( X \) because: (1) there exists no \( p \in (0, \infty) \) such that \( \left( E_x \left[ |X - m_p|^p \right] \right)^{\frac{1}{p}} < \infty \) for all \( X \) (as can be seen by letting \( X \sim \text{Pareto-Lomax}(\lambda \leq p, \xi) \) for any fixed \( p \)); and (2) \( \lim_{p \to 0} \left( E_x \left[ |X - m_p|^p \right] \right)^{\frac{1}{p}} = \text{geometric mean of } |X - \text{mode}| = 0 \) for all continuous \( X \).

3. The Extended Coefficient of Variation (ECV)

A simple alternative to the power function is the inverted (negative) cosine function, \( g(y) = -\cos(\omega y) \), for \( \omega \in [0, \infty) \). This family is attractive both because

\[
E_x \left[ -\cos(\omega(X - m_\omega)) \right] < \infty \text{ for all random variables } X \text{ and frequencies } \omega > 0 , \text{ and because the}
\]
cosine function provides the symmetric bases of the Fourier approximation to the PDF in much the same way as the power function with even integer exponents provides the symmetric bases of the Taylor approximation. We therefore define the extended coefficient of variation (ECV) as

\[ ECV_\omega = \frac{1}{\omega} \cos^{-1} \left( -E_{\omega} \left[ -\cos \left( \omega (X - m_\omega) \right) \right] \right) = \frac{1}{\omega} \cos^{-1} \left( E_{\omega} \left[ \cos \left( \omega (X - m_\omega) \right) \right] \right), \]

where

\[ m_\omega = \arg \inf_{m \in \mathbb{R}} \left\{ \frac{1}{\omega} \cos^{-1} \left( E_{\omega} \left[ \cos \left( \omega (X - m) \right) \right] \right) \right\}. \]

To implement the new risk measure, we must choose a fixed frequency, \( \omega \), at which to calculate it, just as a fixed exponent (e.g., \( p = 2 \)) must be selected for \( CV_p \). For any constant value of \( \omega \), a greater spread of the loss distribution will cause more of the distribution’s probability to lie outside the central “cup” of the cosine-based penalty function. Therefore, it seems intuitively desirable to make \( \omega \) inversely related to the spread of the distribution.

To formalize this process, we prove Theorem 1 below. This result requires consideration of two PDFs of \( \omega \) on the sample space \([0, \omega] \): (1) the noninformative (and therefore improper) prior, \( \nu(\omega) = K \), where \( K \) is an arbitrary positive constant; and (2) the normalized Fourier transform, \( \varphi_x(\omega) \) (or normalized characteristic function, \( \chi_x(\omega) = \bar{\varphi}_x(\omega) \)),

\[ \rho_x(\omega) = |\varphi_x(\omega)|^2 \int_0^\infty |\varphi_x(t)|^2 dt = |\chi_x(\omega)|^2 \int_0^\infty |\chi_x(t)|^2 dt. \]

**Theorem 1:** The frequency \( \omega^* \) that minimizes the marginal Kullback-Leibler divergence (KLD) of \( \nu(\omega) \) from \( \rho_x(\omega) \) also maximizes the marginal Shannon information associated with

\[ |\varphi_x(\omega)|^2. \]
Proof: The KLD of \( v(\omega) \) from \( \rho_x(\omega) \) is given by \( \int_0^\infty \ln \left( \frac{\rho_x(\omega)}{K} \right) \rho_x(\omega) d\omega \). Therefore, the individual frequency \( \omega^* \) that provides the smallest marginal contribution to the KLD is given by

\[
\omega^* = \arg \inf_{\omega \geq 0} \left\{ \ln \left( \frac{\rho_x(\omega)}{K} \right), \rho_x(\omega) \right\}
\]

\[
= \arg \inf_{\omega \geq 0} \left\{ \ln \left( \frac{|\varphi_x(\omega)|^2}{K \int_0^\infty |\varphi_x(t)|^2 dt} \right), \frac{|\varphi_x(\omega)|^2}{\int_0^\infty |\varphi_x(t)|^2 dt} \right\}
\]

\[
= \arg \sup_{\omega \geq 0} \left\{ -\ln \left( |\varphi_x(\omega)|^2 \right) \cdot |\varphi_x(\omega)|^2 \right\},
\]

where \( K \) is set equal to \( \frac{1}{\int_0^\infty |\varphi_x(t)|^2 dt} \) in the last step.

In effect, the theorem shows that if we select the individual frequency \( \omega^* \) to maximize the likelihood of providing a “uniform-like” filtering of the PDF over the frequency domain (and thus attempting to give equal \textit{a priori} consideration to all values of \( \omega \)), then we also select the individual \( \omega^* \) that is most likely to provide useful information about the PDF through its transformation to the frequency domain. As will be seen in the next section, this value of \( \omega^* \) justifies our original intuition by being inversely proportional to the spread of the distribution.

Taking \( \omega^* \) to maximize the marginal Shannon information, the ECV is given by the following theorem.

\textbf{Theorem 2:} If \( \omega^* = \arg \sup_{\omega \geq 0} \left\{ -\ln \left( |\varphi_x(\omega)|^2 \right) \cdot |\varphi_x(\omega)|^2 \right\} \), then

\[
ECV_{\omega^*} = \frac{1}{\omega^*} \cos^{-1} \left( \exp \left( -\frac{1}{2} \right) \right) \left[ \frac{\text{Root}_m \left( E_X \left[ \sin \left( \omega^* (X - m) \right) \right] \right)}{\text{Root}_m \left( E_X \left[ \sin \left( \omega^* (X - m) \right) \right] \right)} \right].
\]
**Proof:** We first consider the denominator, which equals

\[ m_{\omega^*} = \arg \inf_{m \in \mathbb{R}} \left\{ \left(1/\omega^*\right) \cos^{-1} \left( E_X \left[ \cos \left( \omega^* (X - m) \right) \right] \right) \right\}. \]  

Since \( E_X \left[ \cos \left( \omega^* (X - m) \right) \right] \leq 1 \) and \( \cos^{-1}(\cdot) \) is a decreasing function on the interval \([-1,1]\), it follows that \( m_{\omega^*} \) must be given by the value of \( m \) that maximizes \( E_X \left[ \cos \left( \omega^* (X - m) \right) \right] \). Setting the relevant first derivative equal to 0 then implies that \( m_{\omega^*} \) is the value of \( m \) for which \( E_X \left[ \sin \left( \omega^* (X - m) \right) \right] = 0 \).

Now consider the numerator, which equals \( (1/\omega^*) \cos^{-1} \left( E_X \left[ \cos \left( \omega^* (X - m_{\omega^*}) \right) \right] \right) \), where

\[ \omega^* = \arg \sup_{\omega \geq 0} \left\{ -\ln \left( |\varphi_X(\omega)|^2 \right) \right\} \]  

Taking the appropriate first derivative of

\[ -\ln \left( |\varphi_X(\omega)|^2 \right) \]  

and setting it equal to 0 then yields

\[ \ln \left( |\varphi_X(\omega^*)|^2 \right) = -1 \]

\[ \iff |\varphi_X(\omega^*)|^2 = \exp(-1) \]

\[ \iff E_X \left[ \exp(-i\omega^* X) \right] E_X \left[ \exp(i\omega^* X) \right] = \exp(-1) \]

\[ \iff E_X \left[ \exp(-i\omega^* (X - m_{\omega^*})) \right] E_X \left[ \exp(i\omega^* (X - m_{\omega^*})) \right] = \exp(-1) \]

\[ \iff \left( E_X \left[ \cos \left( \omega^* (X - m_{\omega^*}) \right) \right] \right)^2 + \left( E_X \left[ \sin \left( \omega^* (X - m_{\omega^*}) \right) \right] \right)^2 = \exp(-1) \]

\[ \iff E_X \left[ \cos \left( \omega^* (X - m_{\omega^*}) \right) \right] = \exp \left( -\frac{1}{2} \right) , \]

which implies

\[ \frac{1}{\omega} \cos^{-1} \left( E_X \left[ \cos \left( \omega^* (X - m_{\omega^*}) \right) \right] \right) = \frac{1}{\omega} \cos^{-1} \left( \exp \left( -\frac{1}{2} \right) \right) . \]  

\[ \blacksquare \]
4. Properties of the ECV

To explore various properties of $ECV_{\omega}$ in greater detail, we derive analytical expressions for this risk measure for the Lévy-stable family, which is appropriate for modeling the total losses from a large portfolio, and can be used to capture the effects of both skewness and heavy tails.

4.1. The Lévy-Stable Family

The Lévy-stable $(\alpha, \beta, \gamma, \delta)$ distribution is defined on the sample space $(-\infty, \infty)$. Given that its PDF can be expressed analytically in only a few cases (e.g., the Gaussian and Cauchy families), the general distribution is usually described by its characteristic function,

$$\chi_X(\omega) = \exp(-\gamma^\alpha |\omega|^\alpha) \exp\left( i \left[ \delta \omega + \gamma^\alpha |\omega|^\alpha \beta \sgn(\omega) \tan\left( \frac{\pi}{2}(1 - \gamma^{1-\alpha} |\omega|^{-\alpha}) \right) \right] \right).$$

Since there are many possible parameterizations of the Lévy-stable distribution, we note that the selected parameterization is equivalent to $S(\alpha, \beta, \gamma, \delta; 0)$ as defined in Nolan (2008), where:

- $\alpha \in (0, 2]$ is the tail parameter (with smaller values of $\alpha$ implying heavier tails, and $\alpha = 2$ in the Gaussian case);
- $\beta \in [-1, 1]$ is the skewness parameter (with negative [positive] values implying negative [positive] skewness, and $\beta = 0$ in the Gaussian case);
- $\gamma \in (0, \infty)$ is the dispersion parameter (which is proportional to the standard deviation in the Gaussian case – i.e., $\gamma = SD_X[X]/\sqrt{2}$); and
- $\delta \in (-\infty, \infty)$ is the location parameter (which equals the median if $\beta = 0$, and also equals the mean if $\alpha \in (1, 2]$ and $\beta = 0$, as in the Gaussian case).
The Lévy-stable family comprises the set of all asymptotic distributions arising from the generalized central limit theorem. Just as sums of i.i.d. random variables with finite variances tend to the Gaussian distribution as the number of summands increases to infinity, sums of i.i.d. random variables with heavy tails tend to other members of the general family. Specifically, if the tails of a random variable’s PDF diminish as $1/|x|^{p+1}$ (e.g., if $X \sim \text{Pareto-Lomax}(\lambda = p, \xi)$), then sums of this random variable must tend to a Lévy-stable distribution with $\alpha = p$. For these reasons, the Lévy-stable family is suitable for modeling total losses from a large insurance portfolio, regardless of how heavy-tailed the individual loss amounts are.

The following corollary to Theorem 2 provides the analytical form for the ECV in the Lévy-stable case.

**Corollary 1:** If $X \sim \text{Lévy-stable}(\alpha, \beta, \gamma, \delta)$, then

$$ECV_{\omega^*} = \begin{cases} 
\frac{\gamma \sqrt{2} \cos^{-1}\left(\exp\left(-\frac{1}{2}\right)\right)}{\delta + \beta \gamma \tan\left(\frac{\alpha \pi}{2}\right)\left(\frac{\sqrt{2}}{2} - 1\right)} & \text{if } \alpha \neq 1 \\
\frac{\gamma \sqrt{2} \cos^{-1}\left(\exp\left(-\frac{1}{2}\right)\right)}{\delta + \beta \gamma \frac{2\ln(2)}{\pi}} & \text{if } \alpha = 1 
\end{cases}.$$

**Proof:** From Theorem 2, we know that the numerator of $ECV_{\omega^*}$ is given by

$$\left(1/\omega^*\right) \cos^{-1}\left(\exp\left(-1/2\right)\right)$. To solve for $\omega^*$, consider the first-order condition for this quantity from the proof of Theorem 2:

$$\ln\left|\varphi_x(\omega^*)\right|^2 = -1 \iff \varphi_x(\omega^*) = \exp(-1).$$

For the Lévy-stable family,
\[
|\varphi_x(\omega^*)|^2 = |\chi_x(\omega^*)|^2 = \left[ \exp\left(-\gamma^\alpha |\omega^*|^\alpha\right) \right]^2 = \exp\left(-2\gamma^\alpha |\omega^*|^\alpha\right),
\]

from which it follows that

\[
\exp(-1) = \exp\left(-2\gamma^\alpha |\omega^*|^\alpha\right) \iff \omega^* = \frac{1}{\gamma^{\sqrt{2}}}
\]

(because \(\omega^* \in [0, \infty)\)).

Now consider the denominator of \(ECV_{\omega^*}\), which is the value of \(m\) for which

\[
E_x\left[\sin(\omega^*(X - m))\right] = 0.
\]

Writing \(E_x\left[\sin(\omega^*(X - m))\right]\) as the imaginary part of \(\chi_{X-m}(\omega^*)\),

we see that

\[
\exp\left(-\gamma^\alpha |\omega^*|^\alpha\right) \sin\left((\delta - m_{\omega^*})\omega^* + \gamma^\alpha |\omega^*|^\alpha \beta \text{sgn}(\omega^*) \tan\left(\alpha \frac{\pi}{2}\right)(1 - \gamma^{1-\alpha}|\omega^*|^{1-\alpha})\right) = 0,
\]

for \(\alpha \neq 1\), from which it follows that

\[
m_{\omega^*} = \delta + \gamma^\alpha \frac{|\omega^*|^\alpha}{\omega^*} \beta \text{sgn}(\omega^*) \tan\left(\alpha \frac{\pi}{2}\right)(1 - \gamma^{1-\alpha}|\omega^*|^{1-\alpha}).
\]

Substituting \(1/\gamma^{\sqrt{2}}\) for \(\omega^*\) then yields

\[
m_{\omega^*} = \delta + \beta \gamma \tan\left(\alpha \frac{\pi}{2}\right)\left(\frac{\sqrt{2}}{2} - 1\right) \text{ for } \alpha \neq 1
\]

and

\[
m_{\omega^*} = \delta + \beta \gamma \frac{2 \ln(2)}{\pi} \text{ for } \alpha = 1,
\]

where the latter expression is found by taking the appropriate limits.
4.2. Analytical Quantities

Table 1 summarizes the ECV and various associated analytical quantities for both the general Lévy-stable family and the Gaussian and Cauchy sub-families. From this table, one can make several immediate observations:

(1) The information-based frequency, $\omega^*$, is inversely related to the spread of the relevant PDF – specifically, a decreasing function of both dispersion ($\gamma$) and tail heaviness ($1/\alpha$).

(2) The shift, $m_{\omega}$, is an increasing function of both location ($\delta$) and skewness ($\beta$); and also an increasing [decreasing] function of both dispersion ($\gamma$) and tail heaviness ($1/\alpha$) when $\beta$ is positive [negative].

(3) The ECV is a decreasing function of both location ($\delta$) and skewness ($\beta$); a decreasing [increasing] function of dispersion ($\gamma$) when $\delta$ is negative [positive]; and a decreasing [increasing] function of tail heaviness ($1/\alpha$) for sufficiently small [large] values of $\delta$.

(4) The ECV is simply a constant multiple of the CV in the Gaussian case (which is the only member of the Lévy-stable family for which the CV is well defined).
Table 1. Analytical Quantities for the Lévy-Stable Family

<table>
<thead>
<tr>
<th>Distribution</th>
<th>( \omega )</th>
<th>( m_{\omega} )</th>
<th>( ECV_{\omega} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gaussian ((\alpha = 2, \beta = 0,\gamma = \sigma/\sqrt{2}, \delta = \mu))</td>
<td>(1/\sigma)</td>
<td>(\mu)</td>
<td>0.9191(\sigma/\mu)</td>
</tr>
<tr>
<td>Cauchy ((\alpha = 1, \beta = 0, \gamma, \delta))</td>
<td>(1/2\gamma)</td>
<td>(\delta)</td>
<td>1.8382(\gamma/\delta)</td>
</tr>
<tr>
<td>Lévy-Stable ((\alpha, \beta, \gamma, \delta))</td>
<td>(1/\gamma\sqrt{2})</td>
<td>(\delta + \beta\gamma \tan(\alpha\pi/2)[(\sqrt{2}/2) - 1]) if (\alpha \neq 1)</td>
<td>(\gamma\sqrt{2} \cos^{-1}(\exp(-1/2))) if (\alpha \neq 1)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(\delta + \beta\gamma [2\ln(2)/\pi]) if (\alpha = 1)</td>
<td>(\delta + \beta\gamma [2\ln(2)/\pi]) if (\alpha = 1)</td>
</tr>
</tbody>
</table>

The explicit dependence of \( m_{\omega} \) on skewness – noted directly in item (2) and indirectly in item (3) – reveals that the proposed shift is clearly qualitatively different from the mean, which is simply a location parameter. In the case of the CV, setting the shift equal to the mean is sufficient because the associated spread – that is, the SD – is unaffected by skewness. In the case of the ECV, however, skewness effects are embodied in the spread, and therefore also must be incorporated into the shift.

To illustrate the behavior of the ECV as a function of tail heaviness \((1/\alpha)\), we fix the parameters \(\delta = 10\) and \(\gamma = 0.7826\) (to agree with the sum of a portfolio of 10 i.i.d. individual losses with \(E_X[X] = 1.0\) and \(SD_X[X] = 0.35\) in the asymptotic Gaussian case), and plot \(ECV_{\omega}\) versus \(\alpha\) for three distinct values of \(\beta\) (i.e., 0, 0.5, and 1.0)\(^7\) in Figure 1.

\(^7\) We consider only nonnegative values of the skewness parameter because insurance losses are rarely (if ever) negatively skewed.
Consistent with item (3) above, this figure shows that: for any fixed value of $\beta$, the ECV increases as the distribution’s tails become heavier (because $\delta$ is sufficiently large); and for any fixed value of $\alpha$, the ECV decreases as skewness becomes more positive. Thus, for distributions within the Lévy-stable family, the spread-to-shift risk measure of the ECV alerts actuaries to both increased spread from heavy tails and decreased shift from less-positive
skewness. While the latter property may seem somewhat counterintuitive – after all, less-positive skewness in losses is generally thought of as a good thing (in the same way that more-positive skewness in financial gains is thought of as a good thing) – it is in fact entirely appropriate. Just as the conventional CV increases to reflect a decline in expected losses (another good thing), so the ECV must increase to capture the effects of both declining losses and declining skewness.\[^8\]

5. Conclusions

In the present study, we have found that the cosine-based ECV provides a useful spread-to-shift risk measure for insurance losses characterized by substantial skewness and/or heavy tails.

The first step in constructing the ECV is to select an appropriate cosine frequency, \( \omega > 0 \), which is accomplished by maximizing the marginal Shannon information associated with the Fourier transform of the loss random variable’s PDF. In applying this technique to the Lévy-stable family, we found that the optimal value of \( \omega \) increases as the loss distribution becomes more spread out (i.e., from increased dispersion and/or heavier tails). We also observed that the ECV increases as the distribution’s tails become heavier (for sufficiently large values of \( \delta \)), and decreases as its skewness becomes more positive; thus, the new risk measure accounts for increases in a loss’ skewness in much the same way as it accounts for increases in a loss’ location. Finally, we noted that the ECV simplifies to a constant multiple of the CV for the Gaussian family.

\[^8\] Interestingly, this seemingly paradoxical property does not arise when extending the risk/return ratio (given by the inverse of the CV) to skewed asset returns. (See Powers and Powers, 2009.)
In future research, we plan to investigate the Lévy-stable family more closely as a model of total losses from large insurance portfolios, and to develop efficient techniques for estimating its four distinct parameters.
References


